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CALCULATION OF THE MINIMUM DETECTABLE SIGNAL FOR PRACTICAL SPECTRUM ANALYZERS

By
C. Nicholas Pryor

2 August 1971

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13. ABSTRACT

Spectrum analyzers are commonly used in random signal analysis to locate discrete sine wave components in a broadband noise background. This report describes a procedure for determining the sensitivity of practical spectrum analyzers for this task. The minimum detectable signal level for an idealized system is first computed and then adjusted to account for deviations from ideal behavior in a practical system. It is felt that the result allows prediction of sensitivity within about 0.5 decibels.

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Prepared by:
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ABSTRACT: Spectrum analyzers are commonly used in random signal analysis to locate discrete sine wave components in a broadband noise background. This report describes a procedure for determining the sensitivity of practical spectrum analyzers for this task. The minimum detectable signal level for an idealized system is first computed and then adjusted to account for deviations from ideal behavior in a practical system. It is felt that the result allows prediction of sensitivity within about 0.5 decibels.

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Calculation of the Minimum Detectable Signal for Practical Spectrum Analyzers

This report describes a procedure for calculating the performance of spectrum analysis equipment, including the losses due to non-ideal aspects of their practical implementations. The material should be of interest to those engaged in random signal analysis or signal processing system design and evaluation. The work leading to this report was performed in the Electronics and Electromagnetics Division of the Physics Research Department and was performed under task A370-370A/WF08 121 703 Problem 201..

ROBERT ENNIS
Captain, USN
Commander

Z. I. Slawsky

Z. I. SLAWSKY
By direction

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INTRODUCTION

Spectrum analysis is a commonly used technique for evaluating the properties of random time functions. Systems for performing spectral analysis can be divided into two broad categories. One class divides the spectrum into a number of relatively broad bands (such as 1/3 octave filtering) and measures the power in each band to provide an overall description of the spectral power distribution. The second class uses a large number of analysis bands, where the bandwidth of each is a small percentage of its center frequency. These systems are generally referred to as narrow-band spectrum analyzers and are more often used to locate discrete narrow band components of the input which would normally be buried in the overall broadband spectrum.

If the discrete frequency component is considered to be a signal of power S added to a broadband noise background whose power density is N power units per Hertz, it is interesting to determine the minimum signal power which can be distinguished from the background noise. The ratio S/N at which the signal can be distinguished with the desired reliability is referred to as the Minimum Detectable Signal or MDS of the spectrum analyzer. Calculations of this MDS for various systems have been made in a number of ways, using a large variety of source material. Often the basic source of techniques used in these calculations is lost in a long chain of references, with the result that some methods are occasionally misapplied in situations for which they were not originally intended.

The intent of this report is to provide a general approach for analyzing spectrum analyzers of the narrow-band type, based on fundamental or generally accepted principles, and to make all material necessary for MDS prediction available within a single document. The approach is based on determining the MDS for an idealized system, and then adjusting this performance to account for any portions of the actual system which deviate from the idealized behavior. Thus the individual effects of each loss mechanism in the spectrum analyzer may be clearly and independently evaluated. An effort has been made to provide just enough mathematical background to justify each result without overburdening the reader with complicated derivations.

IDEAL MULTI-CHANNEL FILTER SYSTEM

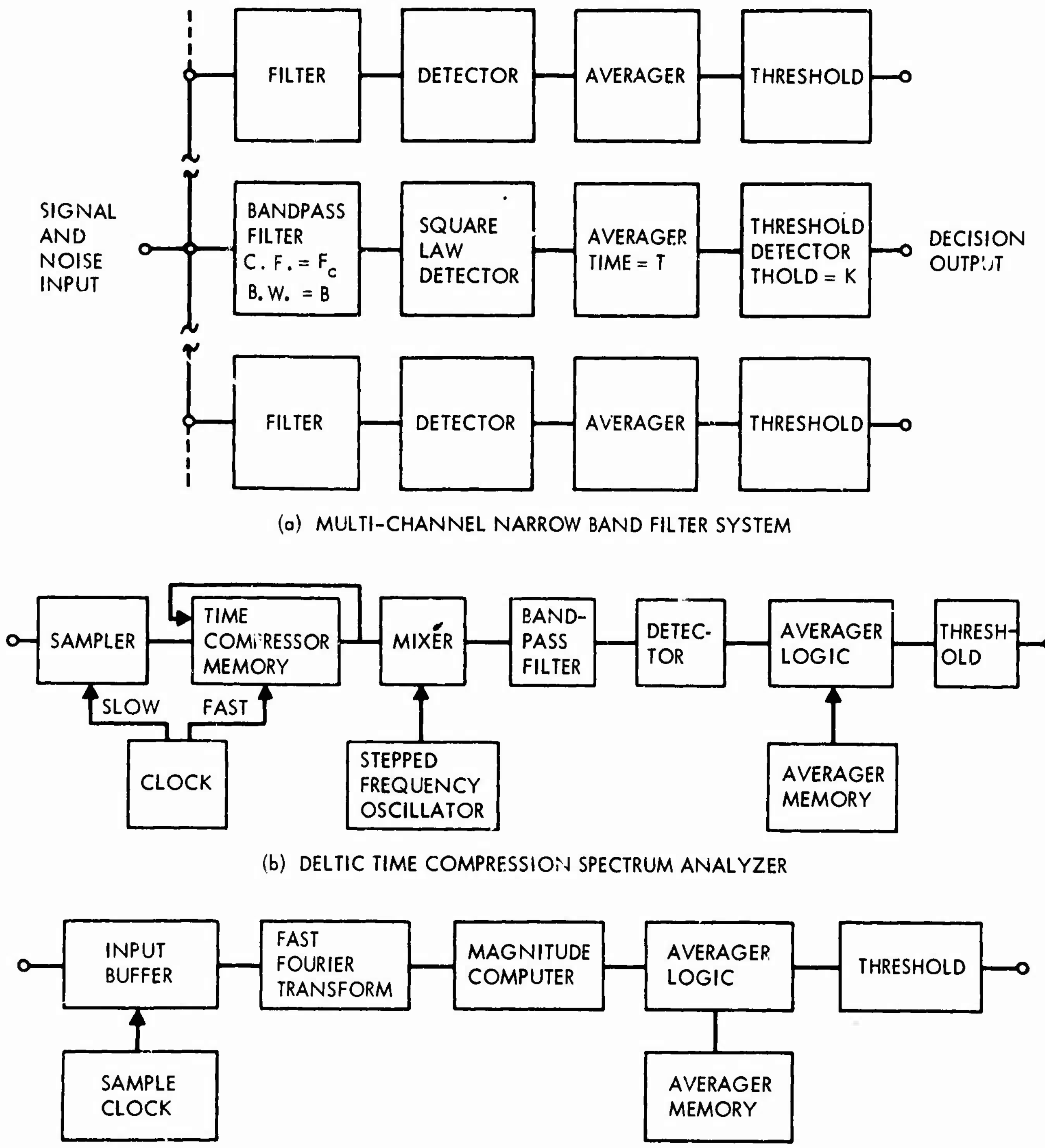
The basic model used for a narrow-band spectrum analyzer is shown in Fig. 1(a) and consists of a bank of narrow-band filter channels in parallel, each tuned to a different frequency in the band to be analyzed. Each channel consists of a narrow band filter of bandwidth B followed by a square-law detector and an averager of integration time T. The filter in the ideal system is assumed to have flat response over the passband B and zero response outside this band. The averager is assumed to compute the running average of the last T seconds of its input, or the function

$$a_o(t) = (1/T) \int_0^T a_i(t-x)\,dx \, .$$

The output of this averager is compared on a continuous basis to a threshold K to determine whether a narrow band signal is present in addition to the broadband noise in the band. It is assumed that some means is available to determine the appropriate value of K. This system (with the exception that the square law detector is only an approximation to a Bessel function) can be shown to be the optimum system for detection of a sine wave in noise.

Practical spectrum analyzers are seldom built in exactly this way, due to the large number of parts involved. One of the common techniques used is shown in Fig. 1(b) and is generally referred to as a time-compression or Deltic approach. The input signal is recorded in a circulating memory and played back repeatedly at much higher speed so that a single analyzer can perform a number of computations on the same piece of input data. The analyzer consists of an oscillator whose frequency can be stepped over the desired range, a mixer, and a single narrow band filter and detector. On successive repetitions of the input data the oscillator frequency is stepped so that a different part of the input signal band is translated into the filter passband. This is equivalent to feeding the same segment of signal sequentially through each of the filter channels of Fig. 1(a). If post-detection integration is desired, an averager memory is updated each time the spectrum analyzer scans through a given frequency cell, and a separate average is maintained for each cell. As each memory word is updated it is compared with the threshold K (which may be different for each cell) to form the detection decision.

Another common technique as shown in Figure 1(c) uses the fast fourier transform (FFT) algorithm to form a complex spectrum of a block of input data that has been accumulated in an input buffer. This spectrum is then converted to magnitude form and used to update all cells of an averager memory. The detection decision is also made each time the averager is updated. Both of these systems are, to a first approximation, equivalent to the multi-channel system of Fig. 1(a) but simply substitute high-speed serial processing for much of the complex parallel equipment. However, these systems introduce a number of approximate techniques such as sampling and quantizing at several points, and they use more practical functions for implementing both the bandpass filter and the averaging integrator than those assumed in the ideal system of Fig. 1(a). In this report the performance of the ideal system is analyzed first, then corrections are made to account for the individual compromises made in practical systems such as the Deltic and FFT analyzers.



(a) MULTI-CHANNEL NARROW BAND FILTER SYSTEM

(b) DELTIC TIME COMPRESSION SPECTRUM ANALYZER

(c) FAST FOURIER TRANSFORM SPECTRUM ANALYZER

FIG. 1 FORMS OF NARROW BAND SPECTRUM ANALYSIS SYSTEMS

IDEAL SYSTEM PERFORMANCE

The performance of this ideal system can be determined by tracing a signal and the broadband noise through each step of the processing. The inputs to the system may be considered to consist of a sinusoidal signal of power S watts and broadband noise whose power density is N watts/Hertz (using a single-sided spectrum consisting of positive real frequencies only). Thus the total noise power in a band up to a cutoff frequency F_o is NF_o watts if the spectrum is flat over the band. This input spectrum is shown in Fig. 2(a).

When this input signal and noise are passed through the narrow band filter, only those components in the passband remain and yield the spectrum shown in Fig. 2(b). This consists of the full signal power S and a noise power of NB watts.

The square law detector has an output whose average value is, by definition, the mean square value (or power) of the signal and noise inputs or S + NB. Superimposed on this average value are fluctuations due to the random nature of the noise inputs. The power density spectrum of these fluctuations is discussed in Appendix A of reference 1, and components are shown to occur in the low frequency region up to twice the filter bandwidth B and in a region of width about 4B near twice the filter center frequency, as shown in Fig. 2(c). By the definition of a narrow-band spectrum analyzer, the center frequency is high compared to the bandwidth, so these components are widely separated.

Two distinct cases must be considered for the integrator, depending on its integration time T. If T is large compared to $1/F_c$ (the inverse of the center frequency), but small compared to 1/B (the inverse of the filter bandwidth), the only function of the filter is to remove the double-frequency fluctuations at the detector output (the dotted portion of Fig. 2(c)) without changing the low-frequency component of the fluctuations. In these cases there is said to be no post-detection averaging. If T is large compared to 1/B then the system is said to include post-detection averaging, and somewhat different techniques are required for computing the performance. These two cases are described separately below.

Case Without Post-Detection Integration

When no post-integration averaging is used (but the double-frequency components are removed) the statistics of the output have been shown to obey a modified chi-square distribution with two degrees of freedom and a variance of

$$\sigma^2 = N^2B^2 + 2SNB \text{ (no post-detection integration).}$$

Selection of the threshold K in the final detector depends on the desired false alarm probability α. Since by definition a false alarm occurs only when no signal is present, the statistics used to select K are those for the case where S equals zero. The distribution is then chi-square and depends only on the single parameter NB. A dimensionless parameter d can now be defined such that the desired false alarm probability α is obtained when the threshold K is set as

$$K = (d+1)NB.$$

The two-degree-of-freedom chi-square distribution has a particularly simple form and allows d to be expressed in terms of α as

$$d = -1 - \ln \alpha .$$

It is now interesting to ask what value of signal power S would be required so that the average input to the threshold detector would be just equal to the threshold K. Since the average is simply S+NB, we have

$$S+NB = K = (d+1)NB$$

or

$$(S/N) = dB .$$

Thus we have an expression for the input signal to noise ratio (expressed as signal power divided by noise power per Hertz) required to, on the average, just equal the required detection threshold. This is not necessarily the point of 50% detection probability since the mean of the modified chi-square distribution is not exactly equal to its median. However, it is sufficiently close for all values of α smaller than about .1 to allow us to interpret this as the approximate signal to noise ratio giving 50% detection probability or the Minimum Detectable Signal (MDS). It is generally convenient to work in decibel notation for MDS calculations, so the previous equation can be rewritten by taking 10 times the log of each side to give

$$(S/N)_{db} = 10 \log B + 10 \log d \qquad (1)$$

(no post-detection integration)

This simple expression for the MDS is the result of recognizing that the threshold depends on the no-signal case alone and of selecting a detection probability P_d of approximately 50%

Case With Post-Detection Integration

When post-detection integration is used, the integrator can be treated as a low-pass filter operating on the low-frequency fluctuations to reduce their variance. If the integration time T appreciably exceeds the inverse bandwidth (1/B) of the narrow-band filter, the calculations involved in finding the output variance are greatly simplified and the fluctuations at the integrator output approach a gaussian amplitude distribution. In reference 1 it is shown that the integration reduces the variance by a factor BT for the ideal narrow-band filter function assumed, so that the output variance becomes

$$\sigma^2 = N^2B/T + 2SN/T \ .$$

We again wish to select a threshold K for the final detector to give the desired false alarm probability α in the no-signal case. Again a parameter d may be defined such that the desired α is obtained when

$$K = NB + d\sigma = NB + d\ N\ \sqrt{B/T} = NB(1+d/\sqrt{BT})$$

where d is the number of standard deviations between the mean cutput of the detector and the decision threshold. This is consistent with the definition of d used in the case without post-detection integration. The relation between α and d may be obtained from tables of the chi-square distribution with 2BT degrees of freedom, since the averager is essentially summing BT samples with two degrees of freedom each.

As BT increases the chi-square distribution rapidly approaches the gaussian which has its mean and median at the same point. Thus the signal required for 50% P_d may be found by setting the average output S+NB equal to K to give

$$S+NB = K = NB + dN\sqrt{B/T}$$

or

$$(S/N) = d\sqrt{B/T}$$

Writing this in decibel form gives the expression

$$(S/N)_{db} = 5 \log B - 5 \log T + 10 \log d \qquad (2)$$

(with post-detection integration)

Comparing equations (1) and (2), we see that both have a portion which is independent of d (and thus independent of α) and one depending only on d and thus on the false alarm probability. If we define the first portion as the basic processor sensitivity or basic MDS of the system, we have

Basic MDS = 10 log B (without post-detection integration)

or

Basic MDS = 5 log B - 5 log T (with post-detection integration)

Figure 3 is provided as a nomograph to simplify computing this basic MDS figure. This is done for systems with post-detection integration by placing a straight edge between the time T along the first scale to the left and the bandwidth B along the third scale. The MDS in decibels relative to the noise in a one Hertz band may then be read at the point the straight edge crosses the second scale. The nomograph also allows reading the time-bandwidth product BT from the scale at the far right. This quantity is of use in later steps. Example A on the nomograph represents a system with a 0.1 Hertz filter bandwidth and an integration time of 180 seconds (3 minutes). The indicated MDS for this system is -16.3 db, and the BT product is about 18.

The nomograph may also be used for systems without post-detection integration by placing one end of the straight edge at the point marked 1 on the BT scale at the right and extending it through the proper bandwidth on the B scale. The intersection with the MDS scale gives the basic MDS for this system. Example B on the nomograph shows this process for a system with B = 0.1 Hz and no post-detection integration. The MDS in this case is - 10 db.

If no further adjustment is made to the system MDS figure, this basic MDS calculation provides a 50% detection probability with a false alarm probability of about 16%. This is generally too high for practical use, so the second portion involving d must be determined. Figure 4 provides a means of finding d for a given false alarm probability if the approximate BT product for the system is known. The left hand scale of Fig. 4 gives the factor d, while the right hand scale is marked in terms of D = 10 log d and thus gives the MDS correction directly. The solid curves in Fig. 4 provide the value of d or D as a function of false alarm probability for four values of BT product and should be used for determining threshold values. The BT = 1 curve is for systems without post-detection averaging and is simply a plot of $-1 - \ln\alpha$. The BT = 10 and BT = 50 curves represent moderate amounts of post-detection averaging and are obtained from the chi-square distribution with 20 and 100 degrees of freedom respectively. The BT = ∞ curve is for systems with very large BT products and is based on the gaussian distribution. Interpolation between these curves should be used for other BT products.

While the approximation that the mean and the median of the output distribution are coincident gives a good value for the MDS with 50% probability of detection for systems with post-detection averaging, direct use of the BT = 1 curve for MDS in systems without post-detection averaging gives detection probabilities generally in the 40% to 45% range due to the skewed distribution. Thus for this case the dotted curve is provided to give the correct MDS value for 50% detection probability.

As an example of the use of this figure, suppose the system discussed in example A for Fig. 3 is to have an allowed false alarm probability of 10^{-4}. Reading from Fig. 4 for a BT product of 18 gives a value of about 4.7 for d or about 6.7 db for D. The threshold K should thus be set 4.7 standard deviations above the mean output of the averager or

$$K = NB\ (1 + 4.7/\sqrt{18}) = 2.11\ NB\ .$$

The corrected MDS for the system is

Basic MDS	-16.3 db	
Δ_{MDS} (FAP)	6.7 db	
Corrected MDS	- 9.6 db	(with post-detection integration)

where Δ_{MDS}(FAP) represents the change in MDS resulting from the selected false alarm probability.

For the system without post-detection integration used as example B in Fig. 3, a 10^{-4} false alarm probability requires a d of about 8.1 as read from the solid B = 1 curve. The threshold is thus set at

$$K = (8.1 + 1)\ NB = 9.1\ NB\ .$$

The dotted curve should be used in this case to obtain an accurate MDS correction for 50% P_d, and it gives a value of 9.4 db for D. Thus the corrected MDS is

Basic MDS	-10.0 db	
Δ_{MDS} (FAP)	9.4 db	
Corrected MDS	- 0.6 db	(without post-detection integration)

The results obtained in this section correspond to the predicted MDS at 50% detection probability for an idealized system with perfectly rectangular filter bandpass characteristics, an integrator with a rectangular impulse response of duration T, and no sampling or quantizing anywhere in the system. All of these are physically impractical to implement. The remaining sections deal with individual variations from this ideal system and represent their effect as a variation in the predicted MDS. Thus a final MDS prediction for a practical system can be obtained by calculating the idealized MDS as above and adding all corrections for system implementation.

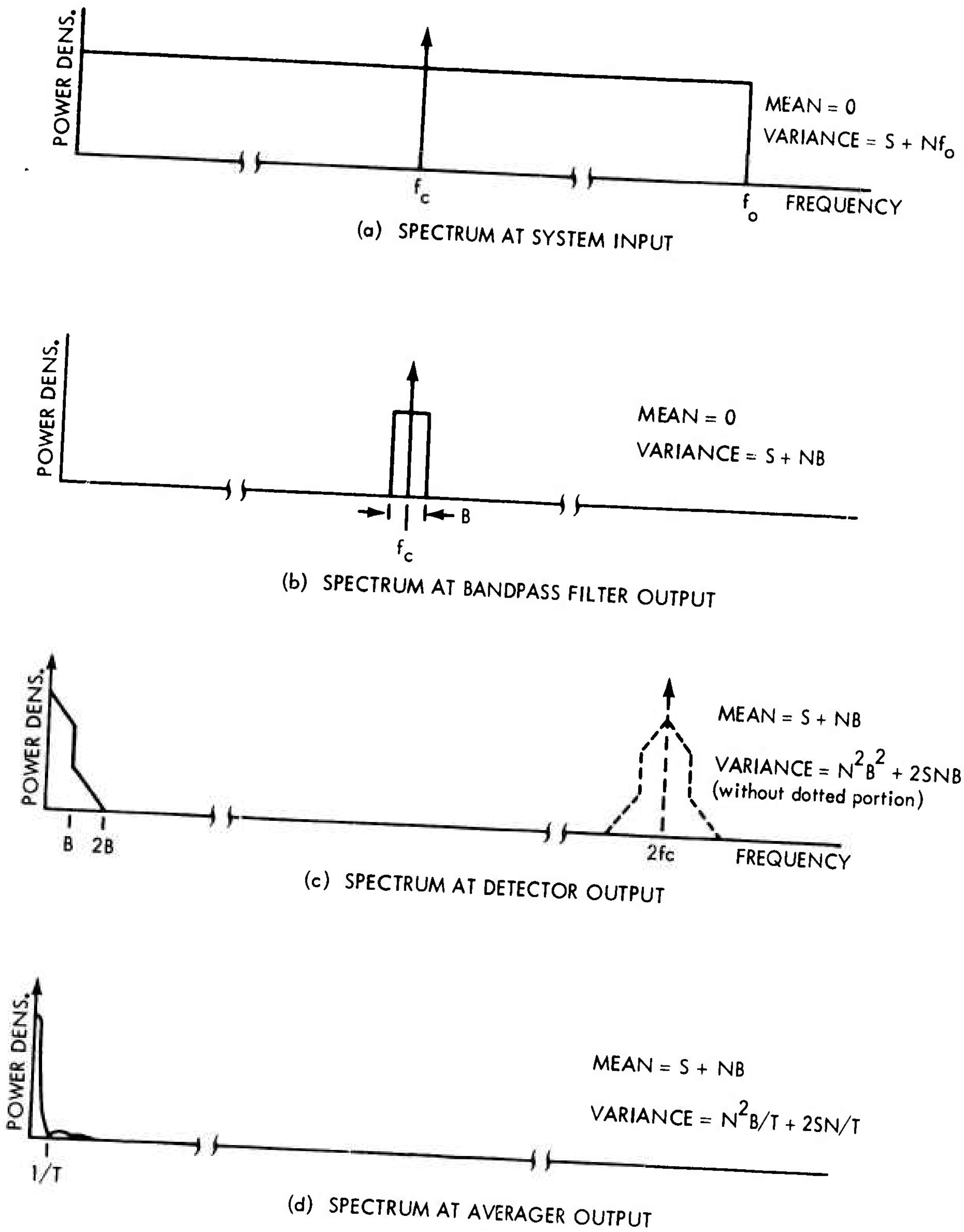


FIG. 2 POWER DENSITY SPECTRA WITHIN NARROW BAND ANALYZER

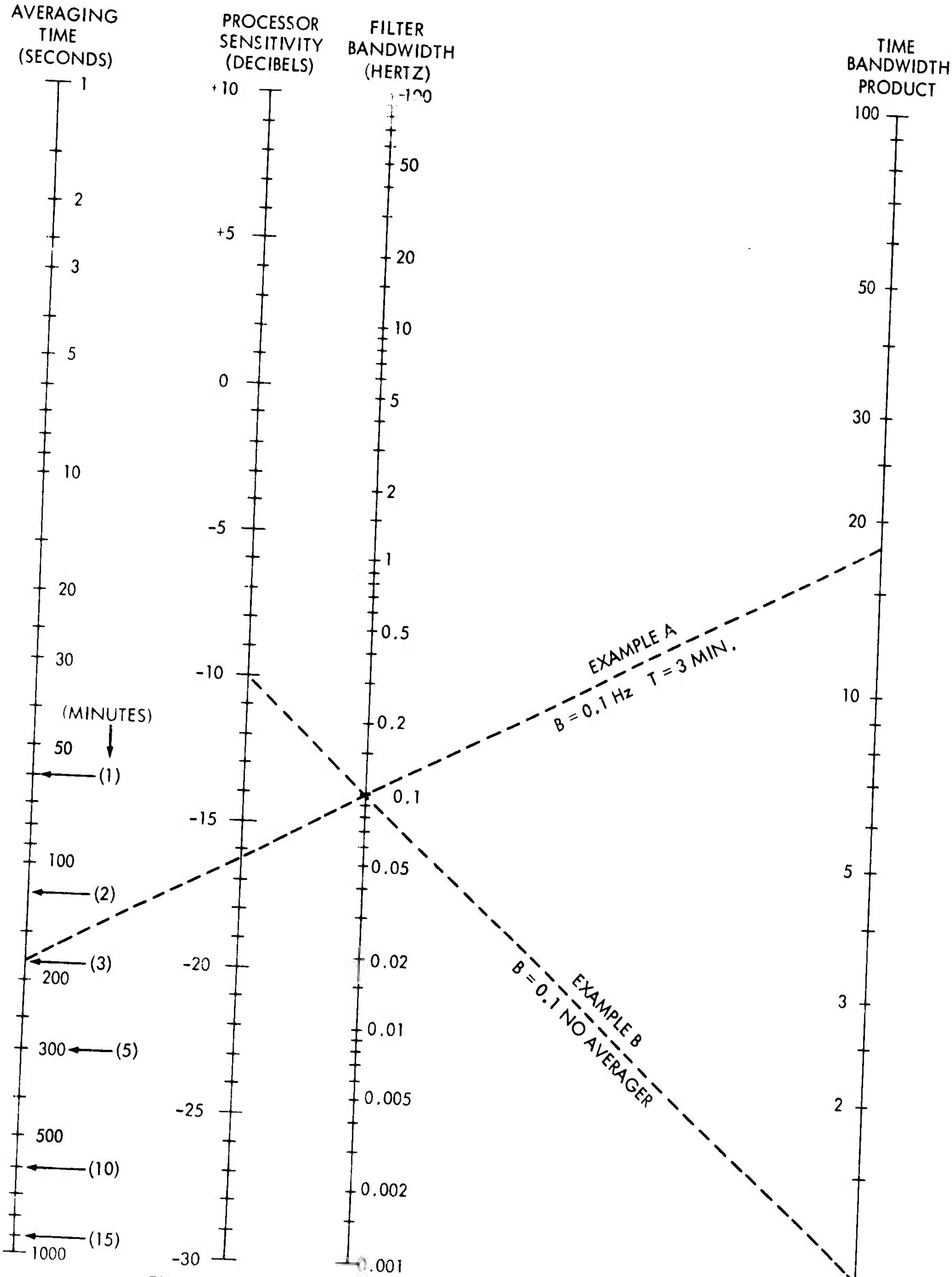


FIG.3 NOMOGRAPH FOR DETERMINING PROCESSOR SENSITIVITY

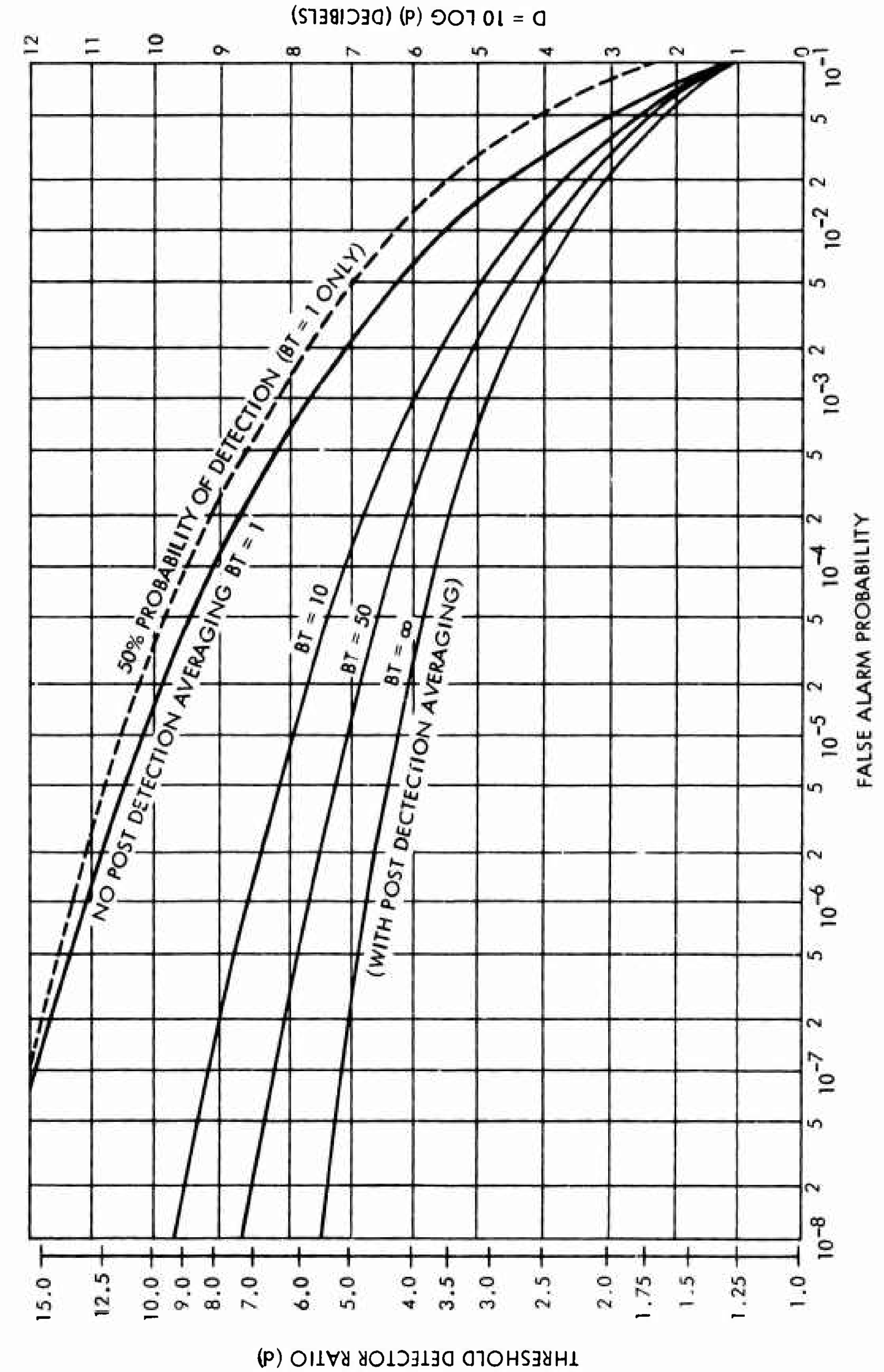


FIG. 4 THRESHOLD DETECTOR LEVEL VS FALSE ALARM PROBABILITY

MODIFIED PROBABILITY OF DETECTION

The simple expression derived for the basic MDS of the spectrum analyzer in the previous section is based on a 50% probability of detection, primarily in order to give a simple form. If some other probability of detection is required, a correction is necessary in the computed MDS level. For systems without post-integration the basic form of the amplitude probability function at the detector output changes when signal is applied, and numerical integration is necessary to evaluate the probability of detection. This has been done by numerous writers, and Fig. 5 is adapted from reference 2 to allow direct determination of D for arbitrary P_d. The dashed line shows an application of this nomograph, extending through the 50% P_d and 10^{-4} false alarm probability points to intersect the D curve at +9.4 db. This confirms the value of D obtained in the previous example from Fig. 4. The dotted curve shows the effect of increasing the required P_d to 95% and maintaining the same false alarm probability. This intersects the D curve at 12.3 db, so the required MDS for the .1 Hz resolution system used in the previous example is now raised to

Basic MDS	-10.0 db	
Δ_{MDS}(FAP & P_d)	+12.3 db	
Corrected MDS	+ 2.3 db	(without post-detection integration)

Note that the MDS adjustment for false alarm probability and detection probability are made together in this case. Note also that the threshold level K depends only on the false alarm probability and is not changed by the new P_d.

The nomograph of Fig. 5 does not apply for systems using post-detection integration, and other techniques must be used to account for detection probabilities different from 50%. As has been shown, the threshold detector on the output of the integrator has a decision threshold of $K = NB + d\,N\sqrt{B/T}$ where d is determined by the false alarm probability. In order to obtain a detection probability different from 50%, the expected output S+NB of the integrator must exceed K by an amount $d'\sigma_s$, where d' is a factor determined by the desired P_d and σ_s is the output standard deviation in the presence of the necessary input signal. The factor d' is positive if P_d exceeds 50% and negative if a P_d below 50% is permitted. The standard deviation of the averager output is given by $\sqrt{N^2B/T + 2SN/T}$ when the signal S is present, so these requirements combine to give

$$S+NB = K + d'\sigma_s = NB + N\sqrt{B/T} + d'\sqrt{N^2B/T + 2SN/T}$$

This may be simplified and written in the form

$$S/N = d\sqrt{B/T}\,[1 + (d'/d)\sqrt{1 + (2/B)(S/N)}]$$

The first portion of this expression is just the MDS for 50% detection probability, so the term in square brackets may be interpreted as a correction factor showing the amount by which the input S/N must be raised to provide the desired P_d. Unfortunately, this term itself depends on S/N, so it is not directly useful as a correction term.

The entire equation above can, however, be rearranged and squared to form a quadratic equation in S/N. Solving this equation gives a form

$$S/N = d\sqrt{B/T}\,[1 + p^2 r + p\sqrt{1 + 2r + p^2 r^2}\,]$$

where

$$p = d'/d \qquad \text{and} \qquad r = d/\sqrt{BT}\ .$$

Again the first portion of the expression for S/N is identical to that for 50% P_d, and the portion in the square brackets is a correction factor accounting for the modified P_d. Since this is a purely multiplicative factor, the correction may be expressed in decibel form as

$$\Delta_{MDS}(P_d) = 10 \log\,[1 + p^2 r + p\sqrt{1 + 2r + p^2 r^2}\,]$$

The parameter p is simply the ratio of the required distances (measured in units of standard deviation) between the mean detector output and the threshold in the signal and no-signal cases. The correction is clearly zero for p = 0 (which is the 50% detection probability case) and is positive for positive p and negative for negative p.

The parameter r also has a physical significance. The factor d may be considered a "signal to noise" ratio at the averager output to give the desired false alarm probability, and $\sqrt{BT}$ is the factor by which the detector output noise is reduced by the averager. Thus r has significance as the detector signal to noise output ratio required to give the desired performance at the averager output. This may be large or small compared to unity, depending on the spectrum analyzer parameters.

Figure 6 shows the MDS change as a function of p and r over the usual range of these parameters. When r approaches zero (meaning very large time bandwidth products in the post-detection averager) the MDS change reduces to 10 log (1+p). However, as seen from Fig. 6, this MDS change increases rather rapidly as the value of r is increased.

Determination of the MDS change for a particular detection probability may be performed by the following steps. First it is necessary to find the value of d' associated with the selected P_d, and this may be done with the aid of Fig. 4. For detection probabilities below 10% simply select the point on the "false alarm probability" scale corresponding to P_d and read the negative of d' from the "threshold detector ratio" scale. The curve corresponding to the approximate time-bandwidth product of the averager should be used. For example, for a 10% probability of detection the value of d' is about -1.28. For detection probabilities above 90%, Fig. 4 may be used by substituting $1-P_d$ for the false alarm probability scale and reading d' directly from the threshold scale. As an example for a detection probability of 95% and a time bandwidth product of about 18 (based on the example in the previous section), d' is about 1.75. This method of determining d' assumes symmetry of the averager output distribution function, which is strictly true only in the gaussian (large BT) case. However, it is a fairly good approximation for systems with finite BT products as well. Figure 4 does not cover values of P_d between 10% and 90%, but a reasonably accurate value can be determined from the linear approximation

$$d' \approx 3.2\ (P_d - 0.5) \qquad .1 < P_d < .9 \quad .$$

Having obtained the value for d' it is now possible to determine the parameters p and r for use with Fig. 6. For the system used in Example A in the previous section the detector threshold ratio d was 4.7 and the BT product was 18. Thus p = d'/d is -1.28/4.7 or -.272 for the 10% detection probability case or 1.75/4.7 = .372 for a 95% P_d. For both cases $r = d/\sqrt{BT}$ is $4.7/\sqrt{18} = 1.1$. Looking at the intersections of these values of p and r in Fig. 6 we obtain an MDS correction of about -2.3 db for the 10% P_d case and about +2.6 db for 95% probability of detection. Thus the -9.6 db figure calculated in example A for a 50% detection probability is lowered to -11.9 if only 10% detection probability is needed or raised to -7.0 db if a 95% P_d is required. Note that for systems with post-detection integration where Fig. 6 is used that the corrections $\Delta_{MDS}(FAP)$ and $\Delta_{MDS}(P_d)$ are determined separately, and both must be added to the basic processor sensitivity to obtain the system MDS.

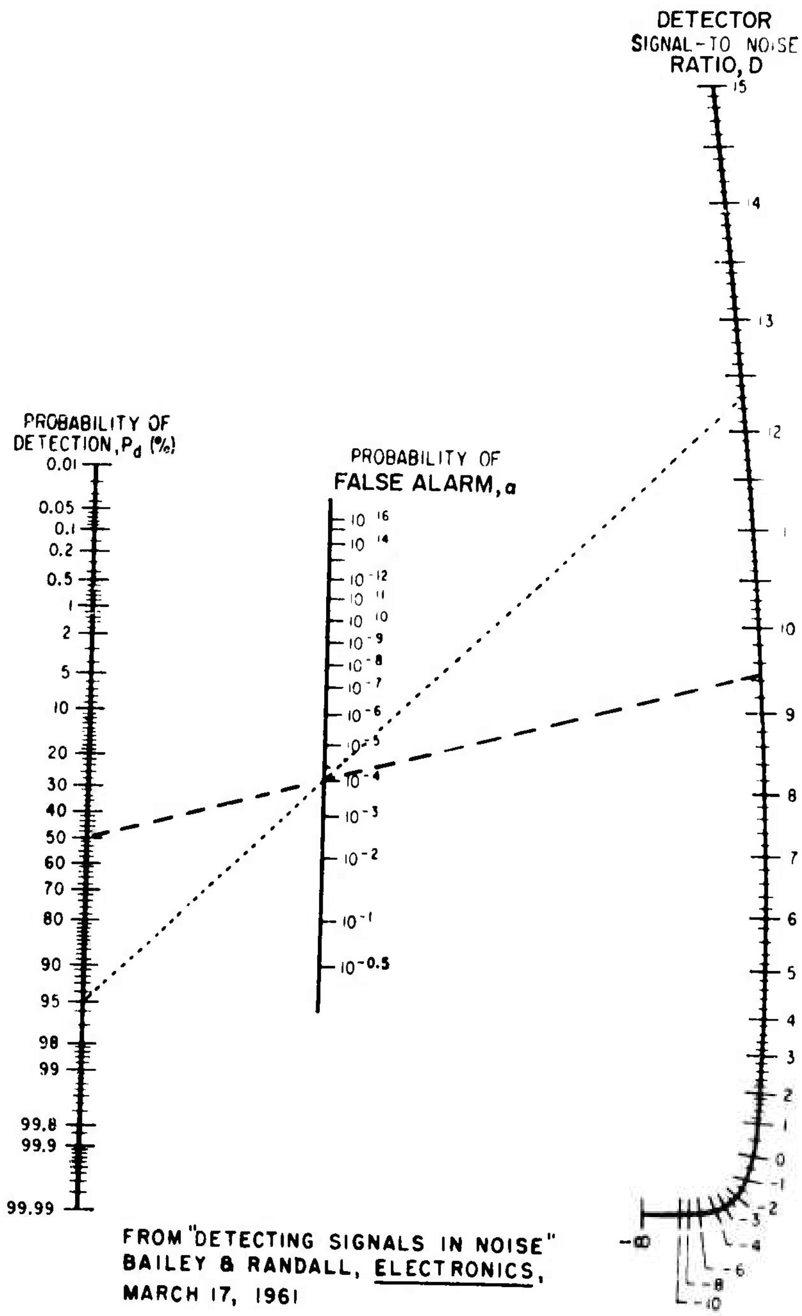


FIG. 5 NOMOGRAPH FOR ARBITRARY P_D WITHOUT POST-DETECTION AVERAGING

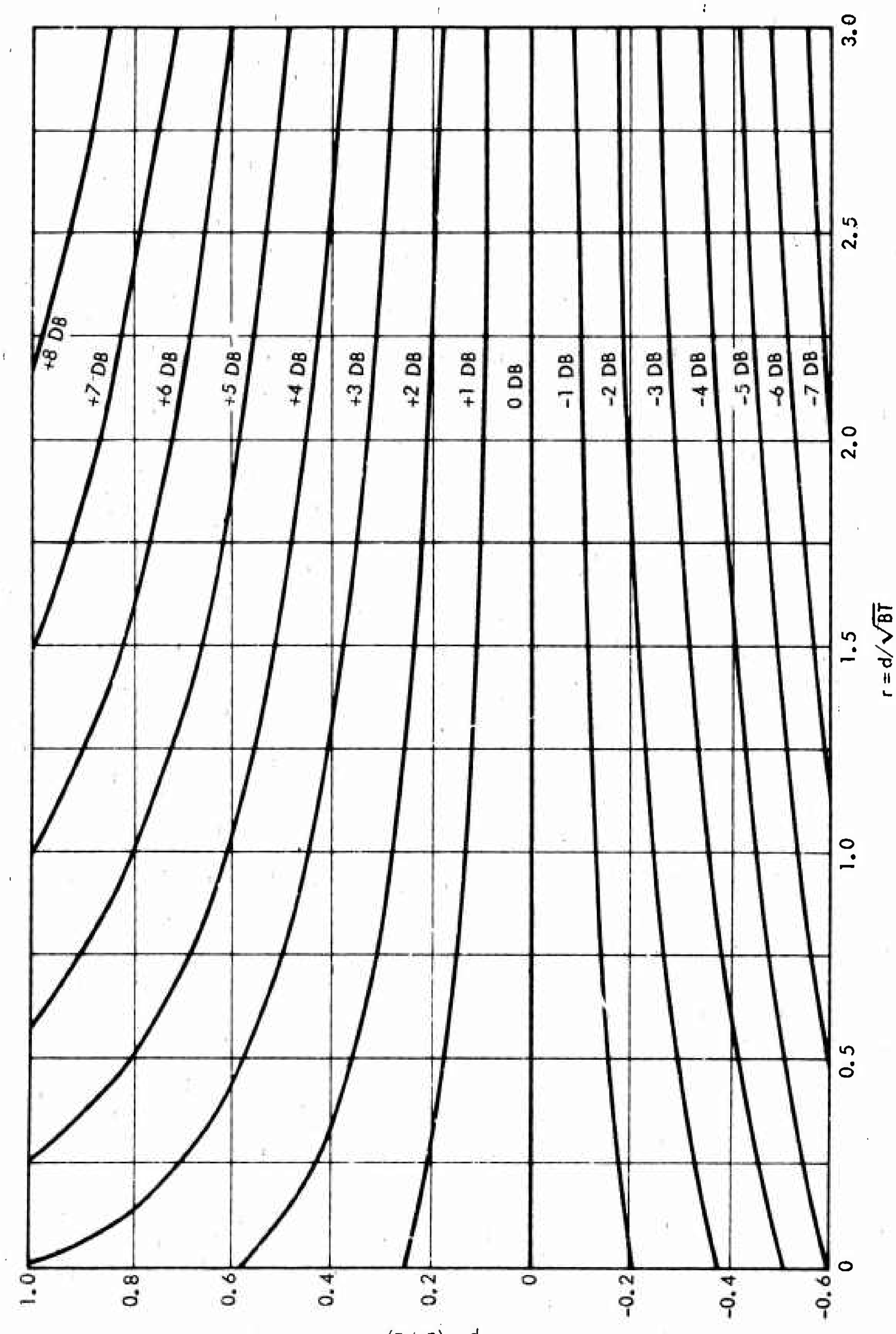


FIG. 6 MDS CHANGE FOR ARBITRARY PROBABILITY OF DETECTION

FINITE DETECTION TIME AND EXPONENTIAL POST-DETECTION INTEGRATION

The idealized post-detection integrator has a rectangular impulse response function which can only be implemented by carrying a complete time history of all detector outputs over the time T. This is generally an unreasonably large amount of information for either digital or analog storage media. A common solution is to approximate the rectangular integrator with a first order recursive filter having a time constant τ approximately equal to T. This is the equivalent of a simple RC low-pass filter on the output of the detector, and it requires only one word of storage per frequency bin to implement in a digital system.

It can be shown (see reference 3) that in the steady state an exponential integrator with time constant τ has a smoothing performance (that is a reduction in the variance at its output) equivalent to a rectangular integration over 2τ seconds. Thus in the steady state the exponential integrator would provide a 1.5 db improvement in MDS over a rectangular integrator if T and τ were made equal. However, the exponential integrator output builds up according to the function $1 - \exp(-t/\tau)$ after the signal is applied, so after any finite interval the signal does not appear at full strength. Since the integrator output change is proportional to signal power, the effective loss in signal strength may be written

$$\Delta_{MDS}(T_d) = -10 \log[1-\exp(-T_d/\tau)] - 1.5$$

where T_d is the time after introduction of the signal by which the detection decision must be made, and the 1.5 term represents the steady-state improvement due to the exponential integrator. This function is plotted in Fig. 7 as a function of the detection time T_d. Similarly, the dashed line is the equivalent increase in MDS required for the rectangular integrator when the detection decision must be made in less than the integration time T. This is given by

$$\Delta_{MDS}(T_d) = 10 \log (T/T_d) \qquad T_d \leq T$$

For the rectangular integrator there is no change in the MDS for any detection time T_d above the integration time T.

Note from Fig. 7 that the rectangular integrator begins to lose performance rapidly when T_d is less than the integration time T, while the exponential integrator follows a comparatively gentle curve. The exponential integrator has a performance poorer than that of the rectangular integrator for detection times between about .7T and 1.25T, and requires allowed detection times greater than about twice the time constant before the predicted steady-state performance is approached. The loss in MDS relative to the idealized system is 0.5 db when the required detection time is equal to the time constant τ.

If the required detection time is known at the time the processor parameters are selected, optimum values for T (for the rectangular integrator) or τ (for the exponential integrator) may be chosen to minimize the MDS. For the rectangular integrator the optimum choice is for T to equal T_d, forming a matched filter for the signal envelope. A loss of 1.5 db in MDS occurs for each factor of two error in matching the integration time to the required detection time. However, for the exponential integrator the optimum tradeoff between increased integration time for more noise smoothing or decreased integration time for faster response may be shown to occur when τ is 0.8 T_d. This optimum, however, is very broad, and the resulting performance varies by less than 0.5 db for a factor of two variation in either direction from the optimum τ.

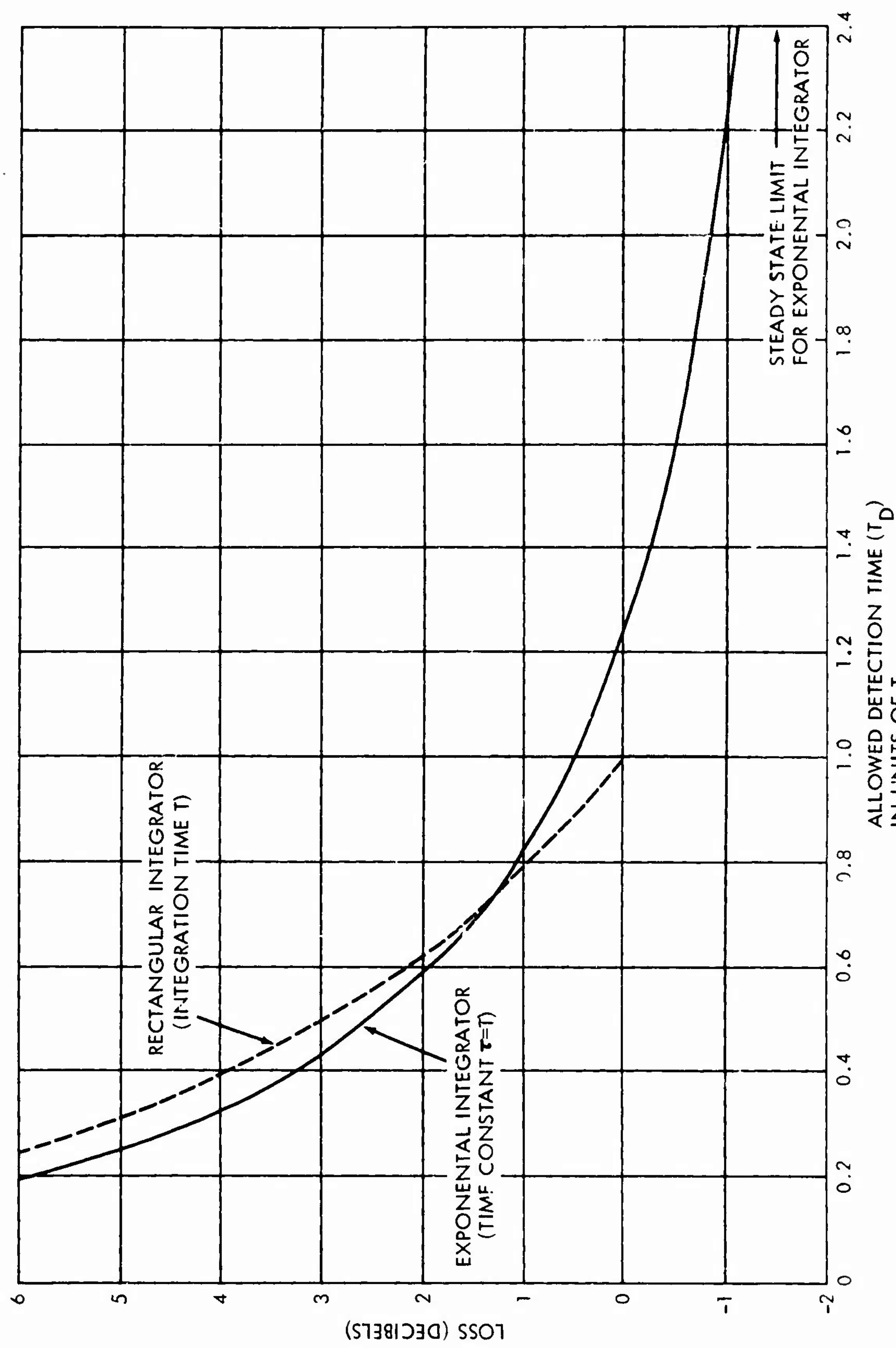


FIG. 7 LOSS DUE TO FINITE DETECTION TIME

PRACTICAL BANDPASS FILTER FUNCTIONS

The bandpass filter in the idealized system was assumed to have perfectly flat response over a band of width B and zero response outside this band, as shown in Fig. 8(a). Practical filters, of course, do not have this type of response, and several representative filter forms are shown in Figs. 8(b) through 8(f). The first two of these are first-order (simple LC resonance) and third-order Butterworth filter forms as commonly used in analog or Deltic type spectrum analyzers. While the Butterworth form was chosen for the third-order filter, it is representative of other types such as Tchebycheff or Papoulis with the same number of poles. Each of these filters has a bandwidth B between its 3 db response points. The next filter has a gaussian amplitude response function, which is an approximation often used in multi-pole analog filters. The nominal bandwidth B was chosen to be the bandwidth to the "1 sigma" points on the response and is <u>not</u> the 3 db bandwidth for this filter. The final two functions are typical of digital systems (such as FFT machines or other transversal filter correlators) where the sample of input signal being correlated is of length 1/B. The first is for a correlator which does not attempt to weight the signal block before correlating, while the second is the response function for systems using Hanning weighting of the data. Notice that the 3 db bandwidth is not equal to B in either case, because of the choice of defining B by the block length.

The use of these modified filter functions has two direct results on the output fluctuations from the spectrum analyzer. First, these filters do not generally have the same "noise bandwidth" as the ideal filter, where the noise bandwidth is the integral of the power response and measures the noise power actually passed through the filter to the detector. Second, the spectrum of the noise at the detector output (similar to Fig. 2(c)) is modified, and this influences the performance of the post-detection integration as described in reference 1. This second factor is important only in systems employing post-detection integration.

The effect of an increased noise bandwidth (NBW) is exactly the same as increasing the input noise power by the same factor and therefore causes a change in the system MDS of

$$\Delta_{MDS}(NBW) = 10 \log \frac{NBW}{B} \quad .$$

It is shown in reference 1 that for a general filter response the variance due to the noise input is reduced by post-integration by a factor $F_n T$ rather than simply BT, where F_n is a function of the filter response shape. The value F_n for the "noise-induced" component is used here since this is the term which directly influences the false alarm probability. This can be referred back to an equivalent change in MDS as

$$\Delta_{MDS}(F_n) = -5 \log(F_n/B) \ .$$

For a system without post-detection integration only the first of these two corrections is needed, while systems employing a post-detection integrator require both corrections. Table 1 is extracted from information derived in reference 1 and shows the noise bandwidth (NBW) and F_n for each of the filter forms of Fig. 8. Also listed is the correction in MDS required for systems both without and with post-detection integrators. The first uses only the noise bandwidth correction, while the second includes both the NBW and F_n corrections.

Notice that the correction for the system without integration is always positive (less sensitive system), since the noise bandwidth of each filter is always at least as great as that of the ideal filter. However, when post-detection integration is included the net correction is sometimes negative, reflecting the fact that the detector output bandwidth increase (and therefore its ability to be smoothed by the integrator) is enough to more than offset the increase in total noise power. Thus, neglecting sampling losses to be discussed later, these filters actually outperform the "ideal" filter with respect to the output fluctuation level from the analyzer.

As an example of the use of these corrections, suppose the band-pass filters in the systems discussed previously were replaced by single tuned LC filters. In the system without post-detection integration only the change in noise bandwidth is considered, and this increases the system MDS by 1.96 db from -0.6 db to about +1.36 db. However, in the system with post detection integration the combined corrections for noise bandwidth and detector output bandwidth give an 0.52 db improvement and thus lower the MDS from -9.6 db to about -10.1 db. This result will be modified later when account is taken of losses due to sampling at the input of the integrator.

As a second example, consider the effect on the MDS of digital implementations of the example systems when the choice is made between unweighted or Hanning weighted processing. If no post-detection integration is used, the unweighted correlator has the same MDS (-0.6 db) as the ideal system, while the Hanning weighted system is degraded by 1.76 db to give +1.16 db as the MDS. In the system with post-detection integration the unweighted correlator gives a potential improvement of 0.88 db for a -10.48 db MDS while the Hanning weighted system has a correction of +0.17 db to give a -9.43 db MDS figure. Again these corrections do not take into account possible sampling losses at the integrator input.

Filter Type	Power Response Function ($dF = \omega/2\pi - f_c$)	3 dB Bandwidth	Noise Bandwidth NBW	F_n	MDS Correction (no post-integrator)	MDS Correction (with post-integrator)
Ideal Bandpass	$1 \quad \lvert dF \rvert < B/2$ $0 \quad \lvert dF \rvert > B/2$	B	B	B	0	0
Single Tuned Circuit	$\frac{(B/2)^2}{(B/2)^2 + dF^2}$	B	1.57 B	3.14 B	+1.96 dB	-.52 dB
Third Order Butterworth	$\frac{(B/2)^6}{(B/2)^6 + dF^6}$	B	1.05 B	1.26 B	+0.2 dB	-.30 dB
Gaussian	$\exp(-(dF/B)^2)$	1.67 B	1.77 B	2.51 B	+2.48 dB	+.48 dB
Unweighted Correlator	$\frac{\sin^2(\pi\, dF/B)}{(\pi\, dF/B)^2}$	0.89 B	B	1.5 B	0	-.88 dB
Hanning Weighted Correlator	$\left[\frac{\sin^2(\pi\, dF/B)}{(\pi\, dF/B)^2}\right] \cdot \left[\frac{1}{(1-(dF/B)^2)^2}\right]$	1.42 B	1.5 B	2.08 B	+1.76 dB	+.17 dB

Table 1. MDS Corrections for Various Bandpass Filter Functions

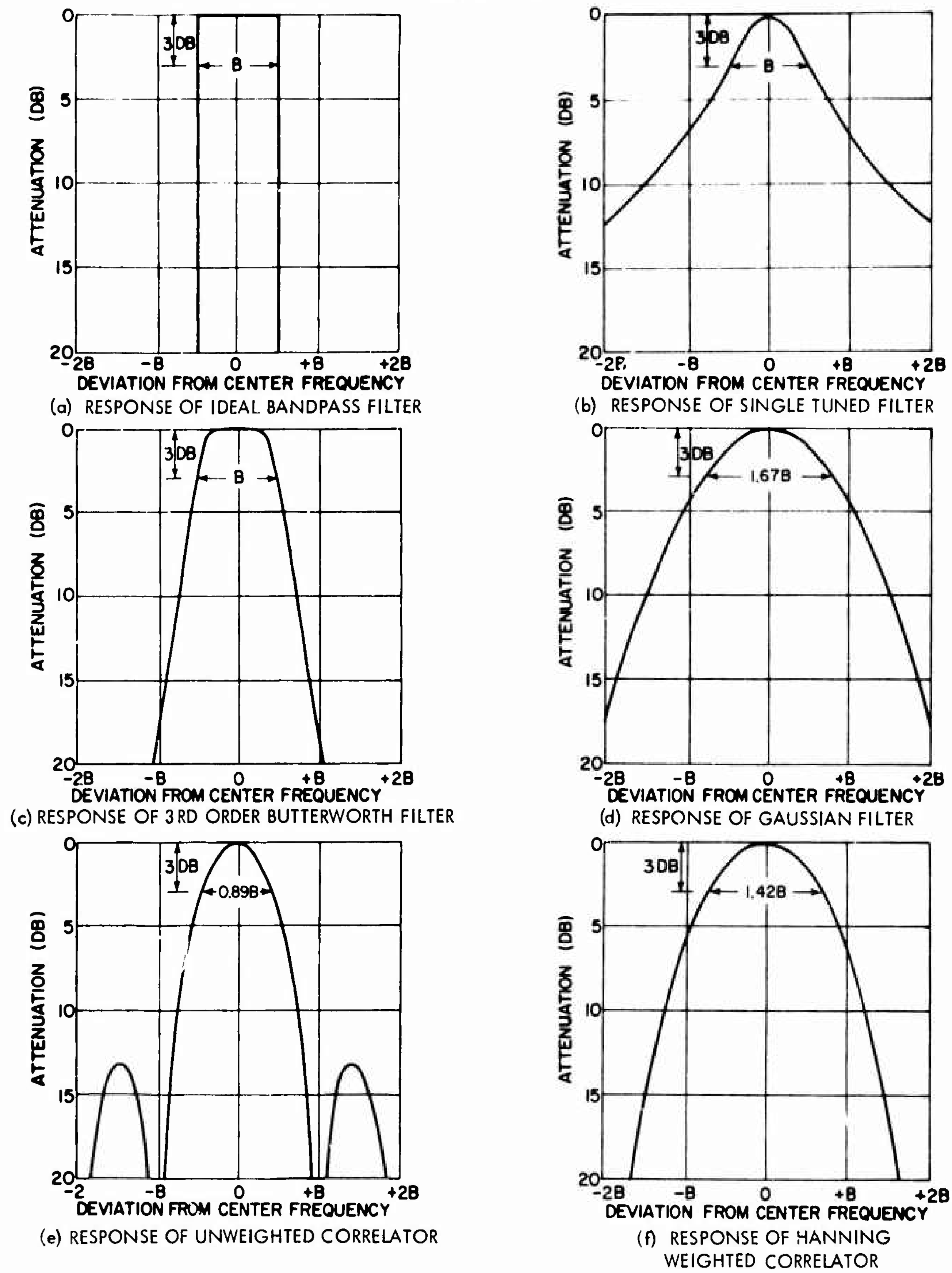


FIG. 8 FREQUENCY RESPONSE OF VARIOUS BANDPASS FILTERS

FILTER SCALLOPING EFFECT

Another effect of using bandpass filters having non-ideal shape characteristics is that the response to a signal input varies as the signal frequency deviates from the exact center of a filter passband. The calculations so far assume that the signal is passed without attenuation by the filter, while in fact, the typical signal is attenuated slightly by not arriving exactly at the peak of the filter response. The effect of this loss of signal on the probability of detection is actually a complicated function depending on many parameters of the spectrum analyzer and must include such things as the increased probability of detection in adjacent analysis bins. However, there are some simpler measures that can give at least an estimate of the system performance degradation.

Suppose the various filter types shown in Fig. 8 are used in a spectrum analyzer where the spacing between adjacent frequency bins is B_o Hertz. The worst-case attenuation of the signal component (which translates directly into the increase in input S/N required to maintain the desired performance for the worst-case choice of signal frequency) can be obtained directly from the filter attenuation at a frequency $B_o/2$ from its center frequency, because any input frequency will lie within $B_o/2$ of the center of some filter response. This is plotted as a function of B_o for each of the filter functions by the solid curves in Fig. 9, and is really just an expanded view of the response curve versus distance from the center frequency.

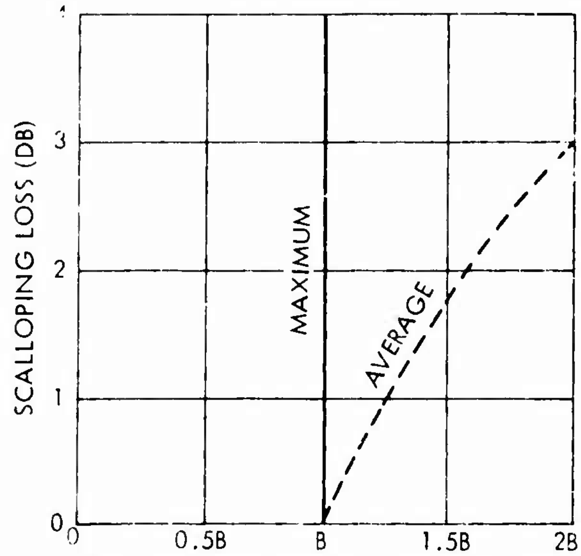
Using the worst-case filter attenuation is, of course, a pessimistic estimate of the scalloping loss for random signal frequencies, since most signals would actually suffer less attenuation through the filter. Thus some sort of measure of the "average" loss is required. Exact determination of the amount by which the input S/N must be increased to maintain the same probability of detection requires at the very least considering the nonlinear character of the P_d versus S/N curve, and even this does not take into account the tendency to detect in more than one analyzer bin. However, a reasonable estimate, which is mathematically tractable, may be obtained by the following argument.

Our original calculation of the MDS was based on setting the average value of the integrator output to the threshold K. We can directly calculate the reduction in the average output due to the signal component over an ensemble of random frequencies by averaging the power response of the bandpass filter over the range within $B_o/2$ of the center frequency. This then represents the amount by which the

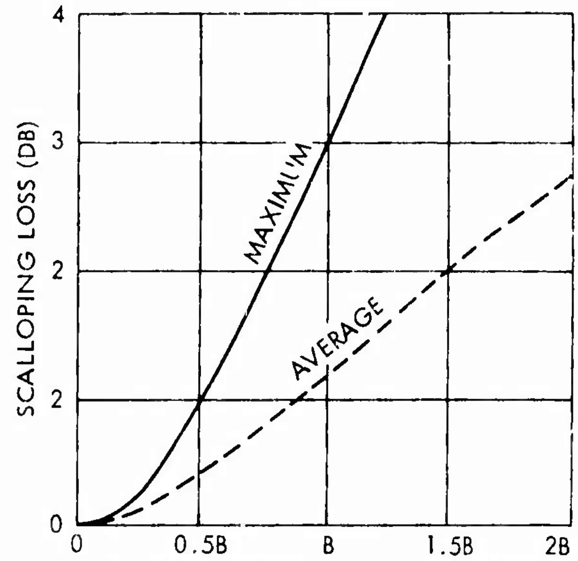
input signal power must be raised so that the average output of the strongest frequency bin is returned to the original level. The MDS correction for this "average" scalloping loss is thus

$$\Delta_{MDS}(\text{scalloping}) = -10 \log(1/B_o) \int_{f_c-B_o/2}^{f_c+B_o/2} |H(\omega-2\pi f_c)|^2 d(\omega/2\pi)$$

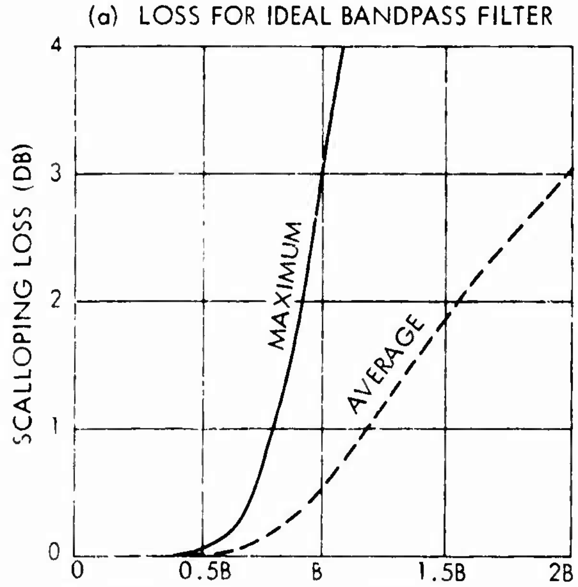
The dotted curves in Fig. 9 show this average scalloping loss as a function of the filter bin spacing. As an example from Fig. 9(c) the worst case loss is 3 db and the average loss is about 0.55 db when the bin spacing B_o is just equal to the 3 db bandwidth B. If the spacing is reduced to 0.8 B (or equivalently if the filter bandwidth is increased to 1.25 B_o) the maximum scalloping loss is reduced to about 1 db and the average loss to about 0.2 db. As another example, Figs. 9(e) and 9(f) show the primary reason for Hanning weighting in digital systems. If the bin spacing is equal to B (or the inverse of the input data block length) the maximum loss is reduced from 3.9 db for the unweighted correlator to 1.4db for the Hanning weighted system. The average loss is similarly reduced from about 1.25 db to about 0.5 db when Hanning weighting is used.



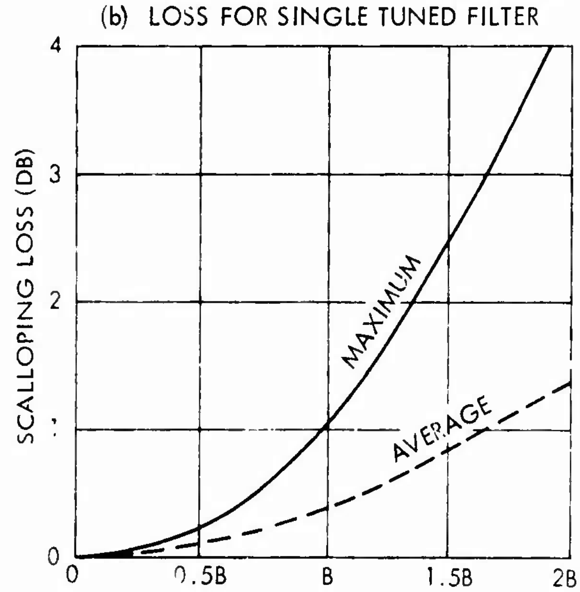
(a) LOSS FOR IDEAL BANDPASS FILTER



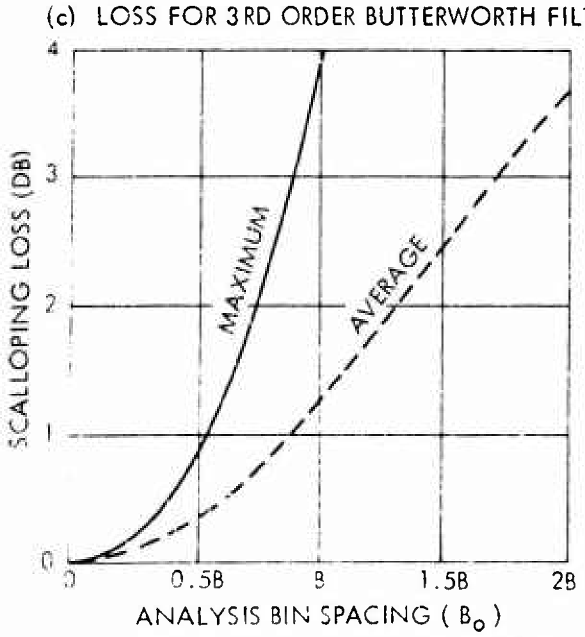
(b) LOSS FOR SINGLE TUNED FILTER



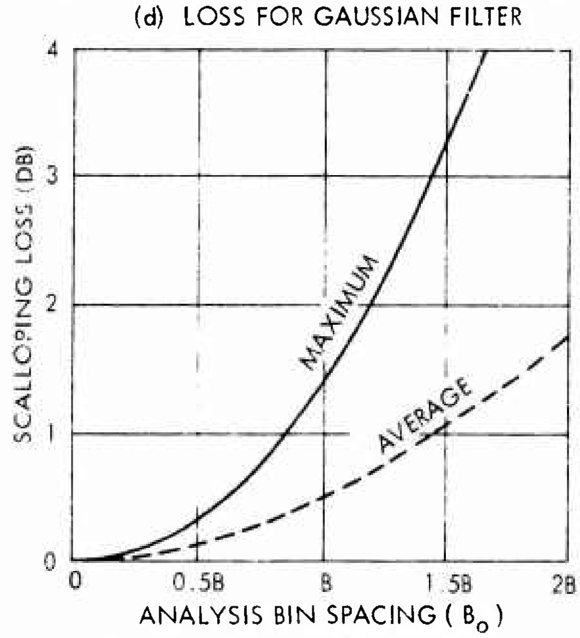
(c) LOSS FOR 3RD ORDER BUTTERWORTH FILTER



(d) LOSS FOR GAUSSIAN FILTER



(e) LOSS FOR UNWEIGHTED CORRELATOR



(f) LOSS FOR HANNING WEIGHTED CORRELATOR

FIG. 9 SCALLOPING LOSS VS BIN SPACING FOR VARIOUS FILTERS

EFFECT OF FILTER TRANSIENT RESPONSE

In the idealized multichannel spectrum analyzer of Figure 1a, each of the frequency bins had its own narrowband filter permanently associated with it. Thus only the steady-state response of the filter to signal and noise inputs was important. However in time compression type systems, such as that shown in Figure 1b, a single filter is effectively switched from one frequency analysis bin to the next on each circulation of the time compressed input. The time spent in each analysis bin is on the order of the inverse of the filter bandwidth, so the transient response of the filter becomes important.

Since the noise in one analysis bin is indistinguishable from the noise in the next, it may be assumed that the filter noise output will be the same as in the steady state case. However when the filter steps through a frequency bin in which a signal is present, the result is equivalent to having the signal applied as a burst with a rectangular pulse envelope of a duration equal to the dwell time in the frequency bin. (Note this assumes that the response of the filter in adjacent frequency bins is negligible and that the heterodyning oscillator changes frequency in discrete steps. If the oscillator sweeps continuously, then the effective envelope of the input signal resembles the frequency response function of the filter and somewhat different results are obtained.)

The output of the filter is applied to the square law detector for measurement of the envelope amplitude, and the detector ouput is then sampled for inclusion in the averager memory. Ideally the sampling instant at the detector output should coincide with the peak of the envelope response to the burst of input signal. However it is common practice to sample at the end of the dwell time in each frequency bin, just as the applied signal component would disappear due to the oscillator switching to the next bin. The envelope response at this instant is just equal to the step response of a low-pass equivalent of the bandpass filter, evaluated at a time T_b equal to the dwell time in the frequency bin. Since failure of the filter to build up to full envelope response during the dwell time is equivalent to reduction of the input signal amplitude by the same ratio, the loss due to this effect is -20 log (envelope) where the envelope response is normalized to unity in the steady state.

The solid curves in Figure 10 show the envelope response of three of the filter functions shown in Figure 8. The time axis represents the dwell time in the bin and is normalized to the nominal

bandwidth B of the bandpass filter. The vertical axis represents the amplitude of the envelope response and is marked in terms of -20 log (envelope) on the right hand end to allow direct reading of the change in MDS resulting from this effect. The single tuned resonant filter and the third-order Butterworth filter are the same as those shown in Figures 8b and 8c with B equal to the separation between the 3 db points on the response. The gaussian filter is not strictly a physically-realizable function. The curve shown in Figure 10 is thus an approximation, chosen to have the bandwidth shown in Figure 8d and a 50% point on its response curve at a time of .5/B. The three dotted curves represent three other filter functions, all of which are third-order (3 pole-pair) filters with 3 db bandwidths of B Hertz and 18 db/octave skirt response for frequencies well removed from resonance. The Tchebycheff filter is allowed one db ripple in the passband (except that it falls to 3 db at ± B/2) while the Papoulis design is ripple-free. Both have steeper skirts near the edge of the passband than the Butterworth design. The "equipole" design consists of three identical sections and has a much less rapid rolloff of response near resonance than the other designs. This design is sometimes used as an approximation to the gaussian filter response.

Notice that the effect of the transient response is a strong function of both filter type and bandwidth when the dwell time and the bandwidth are nearly reciprocal. When T_b is equal to 1/B as is nominally assumed in many spectrum analyzers, the loss among the physically realizable filters ranges from 0.4 db for the single-tuned filter to about 3.5 db for the Techebycheff design. In general the loss increases with increasing steepness of the frequency response curve near the band edge. As an example of the effect of filter bandwidth, consider the Butterworth design which has a loss of 1.3 db when B is chosen as $1/T_b$. If the bandwidth of the filter is increased by 25% so that $B = 1.25/T_b$, the normalized dwell time becomes 1.25/B. In this case the overshoot of the Butterworth filter proves helpful in that the sampled envelope is about 0.1 db larger than the steady state response. On the other hand if the filter bandwidth is reduced to $0.8/T_b$ the loss increases to about 3.8 db.

Since the losses due to the filter transient response can become rather large and are strong functions of the parameters of the spectrum analyzer, care must be taken in applying these results unless the sampling method, the ocsillator stepping method, and the filter functions are exactly as described here. In general the transient response loss may be reduced substantially if a more elegant sampling scheme is employed, and this becomes essential when filters of higner than third order are employed. While this transient response effect does not occur in systems using transversal filters (or correlation techniques) such as FFT systems, it is still important in digital systems using recursive bandpass filter designs.

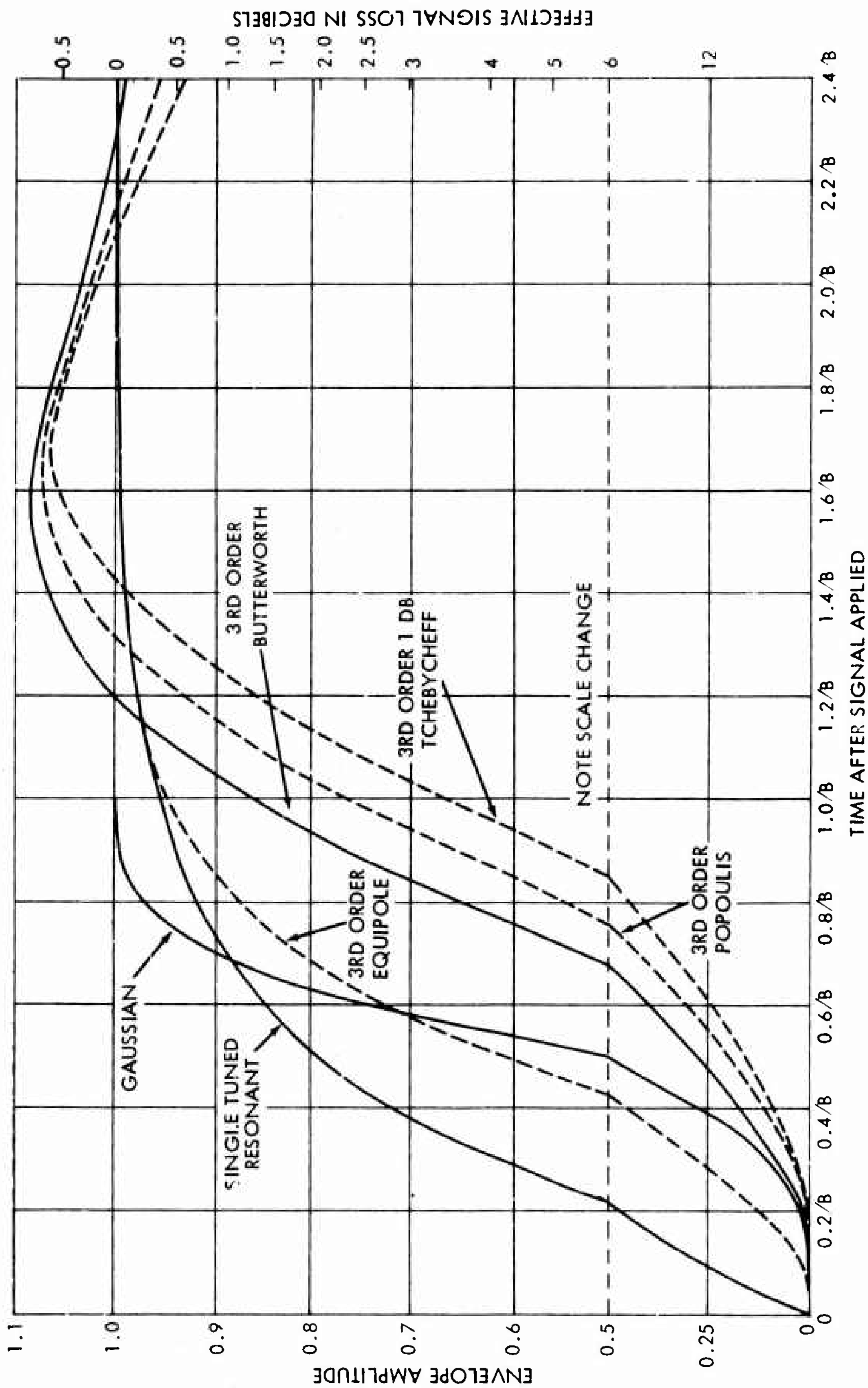


FIG. 10 FILTER ENVELOPE TRANSIENT RESPONSE

INPUT SIGNAL CLIPPING LOSS

Many spectrum analysis systems, particularly those using time compression techniques, hard clip the input signal waveform and preserve only the signal polarity in order to reduce storage requirements in the processor. While this appears to be a rather severe operation to perform on the signal it can be shown that, as long as the noise spectrum is reasonably uniform and no discrete signal frequency component in the input spectrum exceeds the total input noise power, the system degredation due to input clipping is in the vicinity of one decibel. The loss due to clipping can amount to several decibels, however, in exceptional cases such as highly non-white noise or interference between several strong discrete components. This section only covers the case of reasonably white background noise and low input signal to broadband noise ratios.

When the noise spectrum is essentially white over the input band, the clipping operation may be modeled as the addition of a new noise component whose total power is $(\pi/2-1)$ times the total power of the original input noise but whose bandwidth is roughly two or three times the original noise bandwidth. Because of the increased bandwidth of the noise introduced by clipping, the increase in noise power density over the band of interest is smaller than the increase in total noise. However the spectral analysis system almost always employs some sort of discrete input sampling operation and this tends to fold some of the out-of-band clipping noise back into the processing band. Thus the actual loss due to clipping depends to some extent on the input noise spectrum and the frequency at which the clipped information is sampled by the spectrum analyzer.

Reference 4 shows that, if the input noise to a clipper has an autocorrelation function $\phi_{ii}(\tau)$, the effect of the clipper is equivalent to adding a new noise component of autocorrelation function

$$\phi_{ii}(0)\ \sin^{-1}[\phi_{ii}(\tau)/\phi_{ii}(0)]-\phi_{ii}(\tau)$$

For any given input spectrum, the spectrum of this new noise can thus be found. At any given signal frequency in the system passband the effective signal to noise is changed by

$$\Delta_{MDS}(\text{Clipping}) = 10\ \log[(N(\omega)+N_c(\omega))/N(\omega)]$$

where $N(\omega)$ and $N_c(\omega)$ are respectively the noise power densities of the input noise and the noise contributed by the clipper. While the clipping loss can be seen in general to be a function of frequency,

this dependence is small enough in the white (or nearly white) noise case considered here to be ignorable. If the system under consideration employs sampling at a rate less than about five times the highest input frequency, the higher frequency clipper noise components are folded back into the input spectrum and increase the clipping loss. A reasonable assumption here is that the clipping noise is then approximately evenly distributed over a band of width equal to half the sampling rate and that this noise adds incoherently to the original input noise.

Table 2 lists a number of system configurations including both low-pass systems in which several octaves of input signal and noise are passed through a single clipper, and octave bandpass systems in which the analog signal is first filtered to cover only a single octave before clipping. These systems generally have several parallel channels designed to cover multiple octaves. As far as the clipping operation is concerned there is nothing special about dividing the spectrum into octaves rather than broader or narrower segments, but octave filtering is a common choice in many systems because it allows binary interlacing of processing operations in the spectrum analyzer. For both the low-pass case and the octave bandpass case, the input noise is assumed to be approximately white over the system passband and removed outside the passband. For each of the two general input spectra, several typical sampling rates are listed in Table 2.

For both the low-pass and the octave bandpass systems, it may be seen that attempting to sample at the Nyquist rate (twice the input signal bandwidth) after clipping causes a loss of 1.96 db, but that this loss may be reduced significantly by increasing the input sampling frequency. The loss is reduced to 1.09 db by sampling at twice the Nyquist rate. Further increases in sampling rate continue to reduce the loss, but little additional reduction occurs beyond about three times the Nyquist rate.

Processor Input Passband	Sampling Frequency	Clipping Loss
0 to F_o Hz	2 F_o (Nyquist rate)	1.96 db
0 to F_o Hz	3 F_o	1.40 db
0 to F_o Hz	4 F_o	1.09 db
0 to F_o Hz	5 F_o or higher	0.95 db
$F_o/2$ to F_o Hz	F_o (Nyquist rate)	1.96 db
$F_o/2$ to F_o Hz	2 F_o	1.09 db
$F_o/2$ to F_o Hz	3 F_o	0.75 db
$F_o/2$ to F_o Hz	5 F_o or higher	0.65 db

Table 2. Loss Due to Input Signal Clipping

EFFECT OF MODIFIED DETECTOR CHARACTERISTIC

The basic performance calculation for the spectrum analyzer was done for a system employing a square law detector at the bandpass filter output. This was done partly for mathematical convenience and partly because the square law detector has nearly the optimum detector characteristic. However many actual spectrum analysis systems employ other types of detectors. The most common of these is the linear detector (envelope detector, full-wave detector, or half-wave detector) because of its relative ease of implementaion. Another common choice is a logarithmic detector because it produces an output easily interpreted in decibels and because of its "constant-false-alarm-rate" characteristic (output variance is independent of input noise power). A third possibility is a detector which performs a thresholding operation and provides only binary information to further averaging processes.

The effect of these modified detector characteristics on the minimum detectable signal of the spectrum analyzer depends on the signal and noise levels at the detector itself and on the amount of additional averaging done on the detector output. Since the performance depends on detailed characteristics of the tails of the distribution function at the averager output, exact mathematical treatment is quite difficult as may be seen in Reference 5. However reasonable estimates of the MDS change can be made in the two extreme cases of no post-detection averaging and large time-bandwidth product post-detection averaging. The first case can be dispensed with by noting that for any instantaneous output of the square law detector there is a corresponding deterministic output for any alternate type of detector. If the detector output characteristic is monotonically increasing with input power, a threshold K' can be defined such that whenever the square law detector output exceeds a threshold K the alternate detector output exceeds K'. Consequently with proper choice of K' a system with no post-detection averaging will produce detections and false alarms at exactly the same times as a corresponding system using a square-law detector operating on the same input data. Thus if no post-detection averaging is used, modification of the detector characteristic produces no change in the MDS required for a given level of performance.

With large amounts of post-detection averaging, the output distribution of the averager becomes approximately gaussian regardless of the amplitude distribution of the averager input. Under this approximation the distribution is completely specified when its mean and variance are known, and these are fully determined by the mean

and variance of the detector output. Further, if a large amount of post-detection integration is used, then for any reasonable output signal-to-noise ratio requirement the signal-to-noise ratio required at the detector output is small. This means that the detector output variance will still approximately equal the zero-signal variance and that the relationship between the input signal power and the mean output of the detector can be approximated by truncating its Taylor expansion at the linear term. Thus if y represents the detector output whose statistics are a function of the input signal power S, the detector output statistics are represented by

$$\sigma_y^2 = \overline{y^2}(0) - \overline{y}(0)^2 \qquad \text{(variance, at } S=0\text{)}$$

$$\overline{y}(S) = \overline{y}(0) + S \left.\frac{d\overline{y}}{dS}\right|_{S=0} \qquad \text{(mean, versus } S\text{)}$$

or $\sigma_y^2 = C_2 - C_1^2$ and $\overline{y}(S) = C_1 + S\,C_3$

where $C_1 = \overline{y}(0) \qquad C_2 = \overline{y^2}(0) \qquad$ and $\qquad C_3 = (d\overline{y}/dS)_{S=0}$

Now the probability density function of the envelope of the signal plus noise output of the narrow band filter may be shown to be (see reference 6)

$$p(x) = (x/NB) \exp\left[-(x^2+2S)/2NB\right] I_o(x\sqrt{2S}/NB) \quad 0><\infty$$

where S is the signal power, NB is the noise power, x is the envelope amplitude, and I_o is the modified Bessel function of zero order. Now suppose the detector characteristic is such that a given envelope amplitude x produces an output $y = f(x)$. Now the expected value of any function of x may be obtained by multiplying that function by p(x) and integrating with respect to x. Thus the two moments of y which are required may be found as

$$C_1 = \overline{y}(0) = \int_0^\infty x\, f(x)\, \exp(-x^2/2NB)dx/NB$$

and

$$C_2 = \overline{y^2}(0) = \int_0^\infty x\, f^2(x)\, \exp(-x^2/2NB)\, dx/NB$$

where use has been made of the fact that $I_o(0) = 1$. The remaining constant C_3 may be found as follows

$$C_3 = d\bar{y}/dS$$

$$= d/dS \left[\exp(-S/NB) \int_0^\infty x \; f(x) \; \exp(-x^2/2NB) \; I_o(x\sqrt{2S}/NB) \; dx/NB \right]_{S=0}$$

Differentiating the product form and carrying the derivitive under the integral sign gives

$$C_3 = \Big\{ \exp(-S/NB) \left[(-1/NB) \int_0^\infty x \; f(x) \; \exp(-x^2/2NB) \; I_o \; (x\sqrt{2S}/NB) \; dx/NB \right.$$

$$\left.\left. + \int_0^\infty x \; f(x) \; \exp(-x^2/2NB) \; d/dS \left[I_o(x\sqrt{2S}/NB) \right] dx/NB \right] \right\}_{S=0}$$

Now the series expansion of $I_o(z)$ is of the form $1 + z^2/4 + \ldots$, so the I_o function appearing in the above expression may be expanded as

$$1 + x^2S/2N^2B^2$$

so that its derivitive with respect to S as S=0 is $x^2/2N^2B^2$. Inserting this in the above expression and evaluating the remainder of the expression for S=0 gives

$$C_3 = (-1/NB) \int_0^\infty x \; f(x) \; \exp(-x^2/2NB) \; dx/NB$$

$$+ (1/2N^2B^2) \int_0^\infty x^3 \; f(x) \; \exp(-x^2/2NB) \; dx/NB$$

$$= (1/2N^2B^2) \int_0^\infty x^3 \; f(x) \; \exp(-x^2/2NB) \; dx/NB - C_1/NB$$

In order to meet a given performance criterion at the averager output, the applied signal power S must be sufficient to shift the mean output by some number of standard deviations. The signal power required to do this is proportional to the ratio of the detector standard deviation to the slope C_3 or to the function

$$\sqrt{C_2 - C_1^2}/C_3 \quad .$$

For the square law detector $f(x) = X^2$ and we may make use of the integral relation

$$A_n = \int_0^\infty x^{2n+1} \; \exp(-x^2/2NB) \; dx/NB = 2^n \; n! \; (NB)^n$$

to give $C_1 = 2(NB)$, $C_2 = 8(NB)^2$, and $C_3 = (1/2N^2B^2)8(NB)^2 - 2NB/NB = 2$. Plugging these values into the expression for the required power causes all constants to cancel, leaving only the factor NB. Thus for an arbitrary detector function f(x) the power required for a given system performance, relative to that required with a square law detector, may be expressed as an increase in decibels equal to

$$\Delta_{MDS}\text{ (detector)} = 10 \log \left[\sqrt{C_2 - C_1^2}/C_3 NB\right]$$

where C_1, C_2, and C_3 are evaluated for the detector function in question.

It must be remembered that this result applies only for systems in which a large amount of post-detection integration is used, so that both the small-signal approximation to the change in detector output mean and the gaussian approximation to the averager output distribution are valid. Attempts to extend this "mean and variance" approach to systems with small BT products without taking into account the detailed statistics at the averager output can produce misleading results.

The following subsections discuss this result as applied to several commonly used forms of detector function.

Linear Detector

The linear detector may be any one of the class of detectors whose output varies linearly with the envelope amplitude or such that f(x)=x. The MDS change for this type of detector can be evaluated using integrals of the form

$$B_n = \int_0^\infty x^{2n} \exp(-x^2/2NB)\, dx/NB = 1\cdot3\cdots(2n-1)\sqrt{\pi/2NB}\,(NB)^n$$

This leads to $C_1 = B_1 = \sqrt{\pi NB/2}$, $C_2 = A_1 = 2(NB)$, and

$C_3 = (1/2N^2B^2)\, B_2 - B_1/NB = \sqrt{\pi/8NB}$. Plugging these into the expression

for the MDS change gives

$$\Delta_{MDS}\text{ (detector)} = 10 \log \sqrt{2NB - \pi NB/2}\,/NB\sqrt{\pi/8NB}$$

$$= 10 \log \sqrt{(2-\pi/2)/(\pi/8)} = .193 \text{ db}$$

Thus the conclusion is that a degredation of about .2 db is experienced in the input sensitivity of a spectrum analyzer if the square law detector is replaced by a linear detector.

This result is in agreement with other anlytical methods and particularly that of reference 5 for large time-bandwidth products in the post-detection averager. However for one example shown in reference 5, where the performance of a linear and a square law detector were compared as a function of BT product, this limit was approached only for time-bandwidth products in excess of about 100. In fact for BT products below about 70 the linear detector actually outperformed the square law detector by as much as 0.11 db. This is explained by the fact that the true optimum detector, which follows a $\ln I_0(x)$ function, is better approximated by a linear detector for large signal-to-noise ratios at the detector input than it is by the square law detector. On the other hand the square law detector forms a better approximation when only a low detector signal-to-noise ratio is required, as the case for large integrator BT products. Thus for systems with integrator BT products less than about 100 it is more accurate to assume no degredation for a linear detector and to apply a degredation of about 0.2 db only for systems with BT products in excess of 100.

Logarithmic Detector

When the detector function is $y = \ln(x)$, the detector is referred to as a logarithmic detector. This type of detector has two interesting properties. First, the mean output is proportional to the log of the input power and thus may be interpreted in decibels. Second, regardless of the input noise power level, it may be shown that the output variance around the mean value is a constant. For this reason logarithmic detectors are often used in systems where the "constant false alarm rate" property is desired over a large range of input power levels.

The performance of the logarithmic detector may be evaluated by substituting $\ln(x)$ for $f(x)$ in the integral relations for C_1, C_2 and C_3 and evaluating numerically. This calculation indicates a loss of 1.08 db for the logarithmic detector for large integrator time bandwidth products. The theory described here does not suggest the minimum BT product for which this result is valid, but some experimental results suggest degredation on the order of one decibel for BT products as low as ten.

Modified Logarithmic Detector

One of the primary reasons for the degredation in the logarithmic detector is the large fluctuation of $\ln(x)$ for very small x. To avoid this the log detector is often approximated by a linear function x for x less than $\sqrt{NB}$ and by $\sqrt{NB}\,[1+\ln(x/\sqrt{NB})]$ for x greater than $\sqrt{NB}$. This reduces the fluctuations for small x but destroys the "constant false alarm rate" property since the detector shape function

is now tailored to a specific value of NB. Again the values of C_1, C_2 and C_3 may be computed by numerical integration. The indicated degredation in MDS relative to a square law detector is 0.58 db for this modified characteristic, or about half that experienced with the true logarithmic detector.

Binary Detector

The binary detector is one in which an inital thresholding operation is performed before integration, so that only logical "0" and "1" inputs must be handled by the integrator. Since thresholding operations take place both before and after the post-detection integrator, this scheme is sometimes referred to as double-threshold detection. Systems in which the detector output is printed as intensity modulation of a sweep on a paper or CRT display and which then rely on visual integration of many such sweeps for the post-detection integration often result in a form of binary detection. This occurs whenever the dynamic range of the display medium is inadequate to handle the large fluctuations of the detector output and yet provide discernable changes in its average intensity for weak signal inputs.

As was the case with the modified logarithmic detector, the binary detector characteristic must be scaled to the expected detector power input NB. If the detector output is zero for x less than a threshold $k'\sqrt{NB}$ and unity above this value, the degredation due to the binary detector depends on the value of k' selected. For large BT products in the post-detection integrator, a broad minimum occurs in the computed loss for a k' of 1.8 where the false alarm probability at the first detector output is about 20% and the loss due to binary detection is about 0.94 db. The loss increases to 1.02 db for a k' of 2.0 where the false alarm probability at the first detector output is 13.5% and to 1.08 db for a k' of 1.5 where the first detector has a false alarm probability of 32%. Beyond this range the loss increases rather rapidly, reaching 1.74 db at a k' of 2.5 where the initial FAP is 4.3% and reaching 2.07 db at a k' of 1.0 where the initial detector has a FAP of 61%.

Since the binomial distribution resulting in the averager with a binary detector is amenable to analysis, this system has been studied for small BT products and is discussed in reference 7. While the optimum choice of k' varies somewhat with BT product, the loss due to binary detection remains pretty stable at about one decibel even for BT products less than ten. Thus this forms a reasonable estimate for the MDS change due to binary detection, or in many cases to visual post-detection integration.

Modified Binary Detector

One possible modification to the binary detector is a detector whose output is zero up to the threshold $k'\sqrt{NB}$ and proportional to $x - k'\sqrt{NB}$ for all values of x above this threshold. This essentially

corresponds to certain "clean display" techniques in visual integration systems and can reduce computational load in electronic post-detection integration systems. As an example of the performance of this system, using the modified characteristic with a k' of 1.8 reduces the detector loss to 0.53 db as compared to the true binary detector with a loss of 0.94 db.

AVERAGER INPUT SAMPLING RATE

The calculation of processing gain due to smoothing in the averager or integrator assumed that the output of the detector was a continuous signal and constantly available to the averager input. However, in either time compression systems or digital FFT analyzers the same detector is essentially shared among many processing channels, and thus the averager input becomes a sampled time function. In time compression systems the sampling rate is equal to the number of times the oscillator sweeps through each frequency bin per second. There is no particular constraint on the relationship between this and the filter bandwidth, time compressor memory length, or bin spacing, but the sweep rate is generally chosen to be roughly equal to the nominal filter bandwidth. The product of the sweep rate and the time compressor memory length is often termed the redundancy of the processor, since it represents the number of times a single sample of input signal is used to form the output in a given frequency bin.

In fast-fourier transform systems, several of the processing parameters are inherently related by the FFT algorithm. Normally only one transform is done on each block of input data, so that the nominal filter bandwidth, the inverse of the block length, and the sampling rate into the post-detection integrator are all equal. However it is possible to overlap input data blocks so that each input sample typically appears in more than one transform, and thus increase the sampling rate at the integrator input.

The output spectrum from the square law detector was shown in Fig. 2(c) for the ideal bandpass filter and extended up to 2B Hz. For other filter bandpass functions this spectrum does not have a sharp cutoff and thus extends over all frequencies. Consequently, any finite sampling rate at this point results in some information loss and therefore a loss in system performance. This question is explored in detail in reference 1 for each of the filter functions shown in Fig. 8. For any finite actual sampling rate at the integrator input a sampling efficiency is defined which relates the actual output variance of the integrator to that which would be obtained without sampling. Reference 1 distinguishes between a component due only to the input noise, which represents the total fluctuations in the no-signal case, and a second component which appears only when signal is present. Since our MDS computations for 50% probability of detection do not require knowledge of the variance with signal, we need only consider the noise-induced component.

From reference 1 it may be seen that any reduction of this sampling efficiency E_n below unity is equivalent to a similar reduction in the integration time T. Thus the effect of the sampling efficiency on the system MDS is of the form

$$\Delta_{MDS}(\text{sampling}) = -5 \log E_n$$

Figure 11 shows these MDS corrections for various filter functions, as converted from the results in reference 1. The loss due to sampling is determined as a function of the integrator sampling rate relative to the nominal bandwidth B of the bandpass filter. Note this is not necessarily the same as the system redundancy, since the filter need not be matched to the memory length in time compression systems.

As an example, consider a time compression system using a 3rd order Butterworth filter with an 0.1 Hz bandwidth, and which sweeps through all analysis bins once per ten seconds. The sampling rate of the integrator is thus equal to B and the sampling loss may be read from Fig. 11 as 0.5 db. If the sweep rate of the system were increased by 25% (to one sweep per 8 seconds), this loss would be reduced to about 0.2 db. On the other hand, if the filter bandwidth were increased 25% without changing the system sweep rate, the sampling rate is then only 0.8 B and the loss due to sampling increases to about 1.0 db.

As a second example consider an FFT system without overlapped transforms so that the sampling rate is equal to the nominal bandwidth B. If no weighting of the signal data block is done, the loss due to sampling is 0.88 db. Note this exactly cancels the gain shown in Table 1 for this filter function, since the samples are completely uncorrelated at this sampling rate. If Hanning weighting is applied to the data blocks to be transformed, the loss is 1.59 db, which again is just the gain shown in Table 1 for the F_n contribution to the MDS in systems with post-detection integration. It may be further seen from Fig. 11 that, whether or not Hanning weighting is used, the loss due to sampling is reduced to about 0.25 db when the sampling rate is twice the nominal filter bandwidth. This may be implemented in an FFT system by computing a new transform each time half of the input data buffer is updated, rather than waiting for a complete block of new input data before transforming.

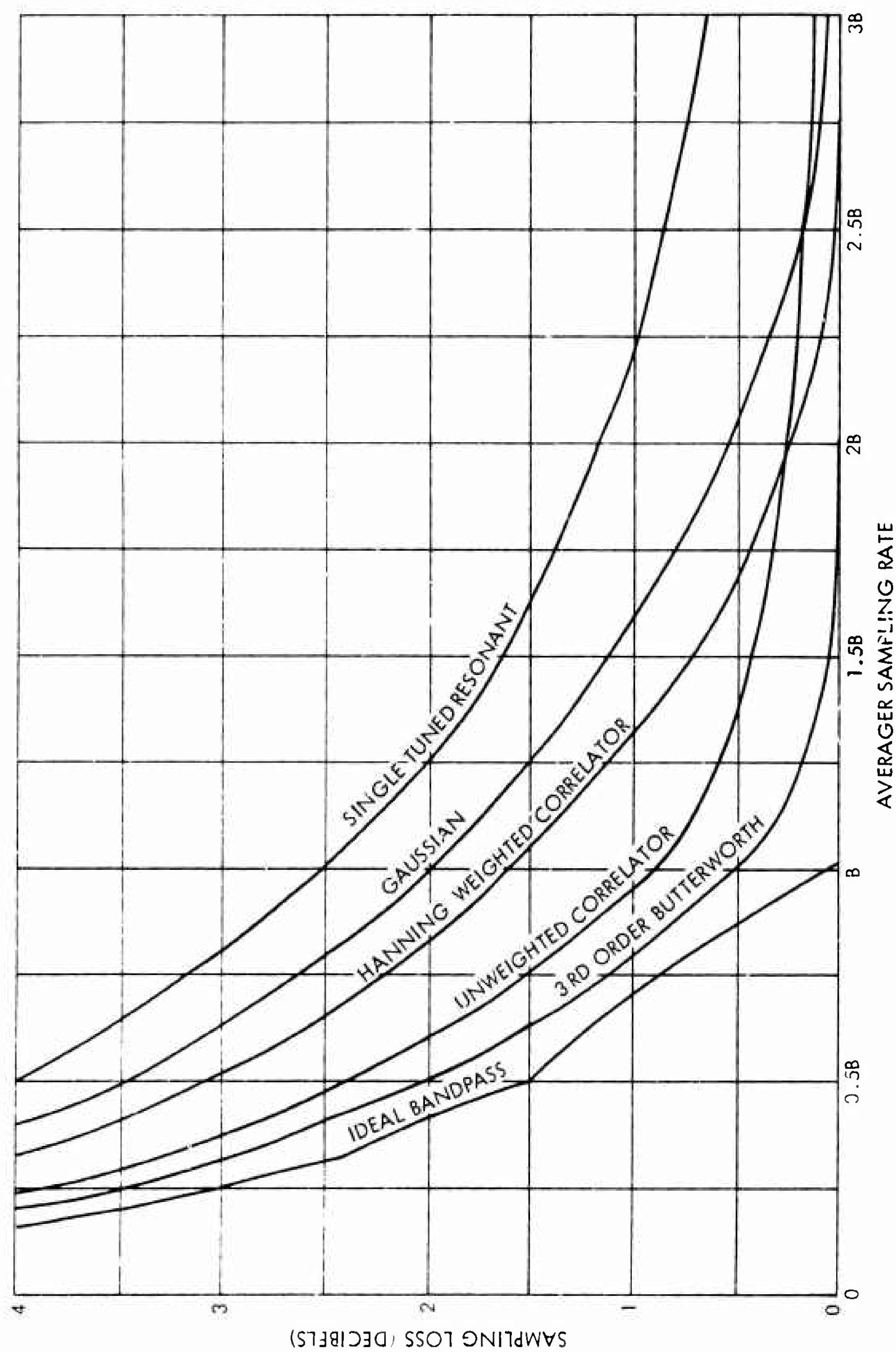


FIG. 11 LOSS DUE TO FINITE AVERAGER SAMPLING RATE

EXAMPLES

In this section the minimum detectable signal is computed for four typical spectrum analyzers. The first two are Deltic type time compression systems, differing only in the choice of the bandpass filter bandwidth. The second two are Fast Fourier Transform systems, and they differ from each other only in that one employs Hanning weighting of the data and the other does not. In all four systems the spacing of analysis frequencies is 0.1 Hz and each frequency output is generated once every ten seconds for inclusion in the post-detection integrators. These integrators in each case are exponential averagers with a three minute time constant, and the output threshold is set to give a 10^{-4} false alarm probability in each frequency bin. A 50% probability of detection is required at an observation time five minutes after a signal is applied. The Deltic systems employ hard clipping of the signal input with a sampling rate equal to three times the highest frequency and use a linear detector, while the FFT systems use multibit signal representation and square law detectors. It is emphasized that Deltic systems do not necessarily employ clipping or linear detectors nor do FFT systems necessarily have multibit signal sampling or square law detectors. These associations are used here for purposes of the examples only. The MDS calculations for these four systems are summarized in Table 3 and are discussed for each system below.

Deltic Spectrum Analyzer

The first step in determining the system MDS is to determine the basic processor sensitivity from Figure 3, using the nominal filter bandwidth of 0.1 Hz and the integration time of three minutes (180 seconds). This gives a processor sensitivity of -16.3 db and shows the BT product of the post-detection integrator to be 18. The first correction to this basic sensitivity is for the false alarm probability of 10^{-4}. Using Figure 4 and interpolating for the approximate BT product gives a Δ_{MDS}(FAP) of about 6.7 db. Since the desired detection probability is 50%, no correction is required here and $\Delta_{MDS}(P_d)$ is zero. Adding these two corrections to the basic processor sensitivity gives an ideal MDS of -9.6 db for the system.

The first variation from this idealized system takes into account the exponential integrator and the allowed observation time. This detection time is 1.67 times the integrator time constant, and Figure 7 shows the resulting MDS is lowered by 0.6 db relative to that for a three-minute rectangular averager. The next correction

is for the filter noise bandwidth and the output bandwidth of the detector fluctuations due to the shape function of the bandpass filter, which is assumed to be a 3rd order Butterworth for this example. Table 1 shows this correction to be -0.3 db. The average scalloping loss for this filter function may be read from Figure 9(c) as 0.55 db, since the nominal filter bandwidth and the frequency spacing are equal. The loss due to the filter transient response is seen from Figure 10 to be 1.3 db, assuming that the Deltic storage time is the inverse of the filter bandwidth or ten seconds.

The loss due to the input signal clipping is estimated from Table 2 as 1.4 db, since the spectrum analyzer is assumed to operate on a low-pass input spectrum and the sampling frequency is three times the input cutoff frequency. The MDS correction for the linear detector is taken to be zero because the BT product of the averager is well below the point where the linear detector becomes worse than the square law detector. Finally from Figure 11 the averager input sampling loss is evaluated to be 0.5 db because the 0.1 per second rate of averager updates is equal to the nominal bandwidth of the Butterworth filter.

The adjusted MDS calculated by adding all these corrections to the ideal system MDS is -6.75 db, or some 2.85 db higher than that of the idealized system. In scanning through the losses it may be observed that much of the loss comes from filter scalloping effects and transient response, both of which may be improved by increasing the filter bandwidth slightly. This is the motivation for the second system.

Modified Bandpass Filter Bandwidth

This system is identical to the previous one except that the bandpass filter bandwidth has been increased to 0.125 Hz in an effort to reduce sampling and transient response losses. Since the nominal value of B has been changed (even though the frequency bin spacing is still 0.1 Hz), a new basic processor sensitivity of -15.8 db is found from Figure 3. The averager BT product is also increased to 22.5, but no significant change occurs in the Δ_{MDS}(FAP) of 6.7 db. Again $\Delta_{MDS}(P_d)$ is zero, so the ideal system MDS is -9.1 db or about 0.5 db worse than that of the previous system.

The corrections for observation time (T_d), filter shape function, clipping, and detector function are unchanged from the previous system since they do not depend on the filter bandwidth. However the average scalloping loss is reduced to 0.2 db, since the frequency bin spacing is now only 0.8 times the filter nominal bandwidth. The transient response effect may be seen from Figure 10 to be a -0.1 db correction, since the dwell time in each frequency bin is increased to 1.25/B due to the increase in B.

Figure 11 however shows that the averager input sampling loss increases to 1.0 db because the averager sampling rate is now only 0.8 B.

While some of the changes due to the increased filter bandwidth tend to improve performance while others tend to degrade it, the adjusted MDS after all corrections are made is -7.5 db. This is 3/4 db better than the system whose filter bandwidth was matched to the frequency bin spacing and Deltic storage time. The list of MDS corrections shows that further improvement in performance would come primarily by eliminating the input clipping signal operation or by increasing the averager input sampling rate. The former requires more storage in the time compressor memories, while the latter requires increasing the time compression ratio (Deltic shift frequency) in order to allow more rapid scanning through the signal spectrum.

Digital FFT Spectrum Analyzer

This system is assumed to accumulate ten-second samples of the input signal and to transform each such sample to form a spectrum with 0.1 Hz frequency spacing. The basic processor sensitivity and the corrections for false alarm probability and detection probability are found in the same way as for the initial Deltic system, to give an ideal system MDS of -9.6 db. Again the five minute allowed observation time on the output of the three minute exponential integrator provides an MDS decrease of about 0.6 db. The correction for the equivalent filter shape function is found from Table 1 to be -0.88 db for the digital correlator which the FFT system resembles. The average scalloping loss for the unweighted correlator is found from Figure 9(e) to be 1.25 db, since the output frequency spacing is equal to the nominal filter bandwidth of 0.1 Hz. The averager input sampling loss is 0.88 db since output transforms are only produced at a rate equal to the nominal bandwidth. No corrections are necessary for filter transient response in correlator type systems, there is no clipping loss, and no correction is required for the square law detector.

The adjusted MDS obtained by adding all corrections to the ideal system MDS is -8.95 db, or just about half a decibel worse than the idealized system. The outstanding loss term is the scalloping loss, which averages 1.25 db and in the worst case amounts to 3.9 db. The fourth system represents an attempt to reduce this scalloping loss.

Digital System with Hanning Weighting

This system is identical to the previous FFT system except that Hanning weighting is employed, either on the signal sample before transforming or on the output spectral estimates, to broaden the equivalent filter shape function. The motivation for this is

similar to that for the second Deltic system. Since the nominal bandwidth is still 0.1 Hz, there is no change in the basic processor sensitivity or Δ_{MDS}(FAP). Similarly the correction for observation time is unchanged. Table 1 shows that the correction for noise bandwidth and detector fluctuation bandwidth is 0.17 db with Hanning weighting, and Figure 9(f) shows that the average scalloping loss is reduced to 0.5 db. Figure 11 shows that because of the increased fluctuation bandwidth with Hanning weighting the averager input sampling loss is increased to 1.59 db.

When all corrections are added to the ideal system MDS, the resultant adjusted MDS is -7.94 db. This is almost exactly one decibel worse than for the system without Hanning weighting, so the conclusion is that Hanning weighting cuases a net degredation even though it substantially reduces the scalloping loss. The reasons for this are the increased noise bandwidth of the equivalent filter with Hanning weighting and the inadequacy of the averager input sampling rate. While the increase in noise bandwidth is inherent in Hanning weighting, the averager input sampling loss can be reduced dramatically by computing transforms twice as often with each transform operation reusing half the data from the previous transform. The averager input sampling rate is thus twice the nominal filter bandwidth, and Figure 11 shows that the loss is reduced to about 0.25 db. This would give an adjusted MDS of -9.28 db, which is within 1/3 decibel of the ideal system MDS.

Table 3. MDS Calculations for Example Spectrum Analyzers

	System 1	System 2	System 3	System 4
	Deltic with .1 Hz Filter	Deltic with .125 Hz Filter	FFT with no weighting	FFT with Hanning Weighting
Basic Processor Sensitivity	-16.3 db	-15.8 db	-16.3 db	-16.3 db
Δ_{MDS}(FAP)	6.7	6.7	6.7	6.7
Δ_{MDS}(P_d)	0.	0.	0.	0.
Ideal System MDS	- 9.6 db	- 9.6 db	- 9.6 db	- 9.6 db
Δ_{MDS}(T_d)	- 0.6	- 0.6	- 0.6	- 0.6
Δ_{MDS}(NBW and F_n)	- 0.3	- 0.3	- 0.88	0.17
Δ_{MDS}(scalloping)	0.55	0.2	1.25	0.5
Δ_{MDS}(transient)	1.3	- 0.1	---	---
Δ_{MDS}(clipping)	1.4	1.4	---	---
Δ_{MDS}(detector)	0.	0.	---	---
Δ_{MDS}(ALI sampling)	0.5	1.0	0.88	1.59
Adjusted MDS	- 6.75 db	- 7.5 db	- 8.95 db	- 7.94 db

SUMMARY

The procedures outlined in this report are intended to allow prediction of the minimum detectable signal for a large class of spectrum analysis systems with a minimum of complex mathematics. While no claim is made that all possible sources of performance degradation have been considered, those discussed here are generally the predominant loss mechanisms. The intent of the organization of the procedure is to make the individual losses apparent so that the reasons for deviations from ideal performance in a given system are made clear. It is recognized that many of these loss mechanisms are not truly independent, so that second order corrections are in principle necessary whenever several loss mechanisms (for example logarithmic detection and undersampled post-detection averager input) are simultaneously present. However these interactions are generally small compared to the first order loss corrections, and it is felt that the steps described here should allow prediction of MDS to witnin about 0.5 db in the majority of spectrum analyzers of the types discussed.

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LIST OF SYMBOLS

Symbol	Meaning	Defined on Page
α	False alarm probability	6
$a_i(t)$	Input to averager	2
$a_o(t)$	Output of averager	2
A_n	Value of definite integral	37
B	Nominal bandwidth of bandpass filter	2
B_n	Value of definite integral	38
B_o	Spacing between adjacent frequency bins	26
C_1	First order of detector output	36
C_2	Second order of detector output	36
C_3	Derivitive of detector output	36
d	Decision threshold ratio for FAP	6
d'	Decision threshold ratio for P_d	14
D	10 log(d)	8
E_n	Sampling Efficiency at Averager Input	43
f(x)	Detector amplitude response function	36
F_c	Center Frequency of bandpass filter	5
F_n	Fluctuation bandwidth of detector output	23
F_o	Cutoff frequency of input noise spectrum	5
FAP	False Alarm Probability	9
$H(\omega)$	Bandpass filter response function	27

Symbols	Meaning	Defined on Page
I_o	Modified Bessel function	36
K	Decision threshold at averager output	2
k'	Threshold ratio for binary detector	40
K'	Decision threshold with modified detector	35
MDS	Minimum Detectable Signal	1
Δ_{MDS}(FAP)	MDS Change for given FAP	9
$\Delta_{MDS}(P_d)$	MDS Change for given P_d	14
Δ_{MDS}(Td)	MDS Change for allowed Td	19
Δ_{MDS}(NBW)	MDS Change for filter NBW	22
Δ_{MDS}(Fn)	MDS Change for filter Fn	23
Δ_{MDS}(scalloping)	MDS Change due to filter scalloping	27
Δ_{MDS}(transient)	MDS Change due to filter transient response	29
Δ_{MDS}(clipping)	MDS change due to clipping	32
Δ_{MDS}(detector)	MDS Change due to detector characteristic	38
Δ_{MDS}(sampling)	MDS Change due to averager input sampling	43
N	Noise power density (single sided)	1
NBW	Noise bandwidth of bandpass filter	22
p	Ratio d'/d	15
p(x)	Probability density of envelope amplitude	36
P_d	Probability of detection	7
r	Ratio $d/\sqrt{BT}$	15
S	Input signal power	1
σ^2	Variarnce of averager output	6
σ_y^2	Variance of detector output	36
T	Integration time of averager	2

Symbols	Meaning	Defined on Page
T_b	Dwell time in single frequency bin	29
T_d	Allowed time for detection	19
τ	Exponential averager time constant	19
x	Envelope amplitude of detector input	36
y	Detector output amplitude	36